NATIONAL UNIVERSITY OF SINGAPORE

PC2132 – Classical Mechanics
(Semester I: AY2013/14)

Examiner: Prof B.-G. Englert

Exam, 3 December 2013

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Write your matric number on the answer book. **Do not write your name.**
2. This examination paper contains **FOUR** questions and comprises **THREE** printed pages.
3. Answer **ALL FOUR** questions for a total of 100 marks.
4. Show all your work in the answer book.
5. For each question, **clearly** indicate what constitutes your final answer.
6. The format of this exam is **Closed Book (with authorized materials)** — Lecture notes for PC2132 and personal notes directly related to the module may be consulted during the test, **but no other printed or written material**.
7. The use of **electronic equipment** of any kind **is not permitted**.

1. One-dimensional motion (25=9+6+10 marks)
A point mass m is moving along the x axis under the influence of the force associated with the potential energy

$$V(x) = E_0 \frac{a^2 x^2}{(x^2+a^2)^2} \quad \text{with} \quad E_0 > 0 \quad \text{and} \quad a > 0.$$

(a) For which energy ranges do you have motion with one, two, or no turning points?

(b) For the oscillatory motion between two turning points, what is the period of small-amplitude oscillations?

(c) Answer the questions of (a) and (b) for $E_0 < 0$.

2. Sun and planet (30=8+4+10+8 marks)
A planet of mass m moves on a Kepler ellipse with major half-axis a, numerical eccentricity ϵ, angular momentum $|\boldsymbol{l}| = m\kappa > 0$, and period $T = \frac{2\pi}{\kappa} a^2 \sqrt{1-\epsilon^2}$.

(a) Express the energy as the sum of the kinetic energy and the potential energy when the planet is in its perihelion.

(b) What does the virial theorem say about the time averages of the kinetic and the potential energy, averaged over one period?

(c) Find the time average of the potential energy by an explicit integration.

(d) Derive Kepler's Third Law by combining (a), (b), and (c).

Hint: The integral $\int_0^{2\pi} \frac{d\alpha}{A + B\cos\alpha} = \frac{2\pi}{\sqrt{A^2 - B^2}}$ for $A > B > 0$ could be useful.

3. Relativistic motion (15=5+10 marks)
The Lagrange function for a particle of mass m in relativistic motion is

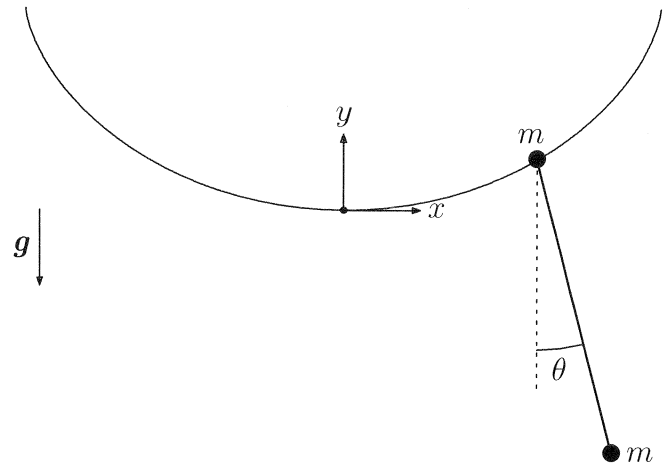
$$L = mc^2 - mc\sqrt{c^2 - \boldsymbol{v}^2} - V(\boldsymbol{r}),$$

where c is the speed of light and $V(\boldsymbol{r})$ is the potential energy.

(a) Show that this is the familiar nonrelativistic expression when $|\boldsymbol{v}| \ll c$.

(b) What is the corresponding Hamilton function?

4. Normal modes (30=10+8+12 marks)



Two equal point masses m are moving without friction in the vertical xy plane, with the gravitational acceleration $\boldsymbol{g} = -g\boldsymbol{e}_y$. The top mass moves along the cycloid parameterized by $(x, y) = R(\phi + \sin\phi, 1 - \cos\phi)$ with $-\pi < \phi < \pi$ and $R > 0$. The bottom mass is connected to the top mass by a massless string of length $3R$; the string is always fully stretched and has angle θ with the vertical direction.

(a) State the Lagrange function $L(\phi, \dot{\phi}, \theta, \dot{\theta})$.

(b) What is the approximate Lagrange function for small-amplitude oscillations around the equilibrium configuration?

(c) Find the normal modes and describe them.

End of Paper