Question 1

The curl of $\vec{F}$ is $\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (m\omega^2\vec{r} - m\vec{\omega}\vec{\omega} \cdot \vec{r}) = m\vec{\omega} \times \vec{\omega} = 0$, so, yes, $\vec{F}$ is conservative, and since $\vec{F}$ is linear in $\vec{r}$, the potential energy is

$$-\frac{1}{2}\vec{r} \cdot \vec{F} = \frac{m}{2}\vec{r} \cdot [\vec{\omega} \times (\vec{\omega} \times \vec{r})] = -\frac{m}{2}(\vec{\omega} \times \vec{r})^2.$$

Question 2(a)

The second derivative of the potential energy is

$$V''(x) = -\frac{d}{dx}F(x) = 2ax,$$

so that $V''(\pm x_0) = \pm 2ax_0$. It follows that there is a stable equilibrium at $x = x_0$ and an unstable equilibrium at $x = -x_0$.

Question 2(b)

For $x \approx x_0$ we have

$$m\left(\frac{d}{dt}\right)^2 (x - x_0) = F[x_0 + (x - x_0)] \approx F'(x_0)(x - x_0)$$

or

$$m\left(\frac{d}{dt}\right)^2 (x - x_0) \approx -2ax_0(x - x_0) = -m\omega^2(x - x_0)$$

with $\omega = \sqrt{\frac{2ax_0}{m}}$ so that the period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{2ax_0}}.$$

Question 2(c)

The period will be longer because it is very long when the energy is close to that required to get to the unstable equilibrium point at $x = -x_0$.

Question 3(a)

From

$$L = \frac{m}{2}\vec{v}^2 - V(\vec{r}) + \vec{v} \cdot \vec{\nabla}u(\vec{r})$$

we have

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial \vec{v}} - \frac{\partial L}{\partial \vec{r}} \\ &= \frac{d}{dt}\left[m\vec{v} + \vec{\nabla}u(\vec{r})\right] - \left[-\vec{\nabla}V(\vec{r}) + \vec{v} \cdot \vec{\nabla}\vec{\nabla}u(\vec{r})\right] \\ &= m\frac{d}{dt}\vec{v} + \vec{v} \cdot \vec{\nabla}\vec{\nabla}u(\vec{r}) + \vec{\nabla}V(\vec{r}) - \vec{v} \cdot \vec{\nabla}\vec{\nabla}u(\vec{r}) \end{aligned}$$

or

$$m\frac{d}{dt}\vec{v} = -\vec{\nabla}V(\vec{r})$$

which doesn't contain any trace of $u(\vec{r})$.

Question 3(b)

The Hamilton function is, for $\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + \vec{\nabla}\,u(\vec{r})$,

$$H = \vec{v}\cdot\frac{\partial L}{\partial \vec{v}} - L\bigg|_{\vec{v}=\frac{\vec{p}-\vec{\nabla}u(\vec{r})}{m}} = \frac{1}{2m}\left[\vec{p} - \vec{\nabla}u(\vec{r})\right]^2 + V(\vec{r}).$$

It depends on the choice of $u(\vec{r})$ because $\vec{\nabla}u$ rules the relation between $\vec{p}$ and $\vec{v}$.

Question 4(a)

The body is composed of a homogeneous ball of radius R and an ellipsoid with half-axes 2R, 2R, R, both having mass density ρ_0. The total mass is, therefore,

$$M = \frac{4\pi}{3}\rho_0 R^3 + \frac{4\pi}{3}\rho_0 4R^3 = \frac{4\pi}{3}\rho_0 5R^3$$

so that

$$\rho_0 = \frac{3}{20\pi}\frac{M}{R^3}.$$

In $\overleftrightarrow{I} = \int (d\vec{r})\,\rho(\vec{r})\left(r^2\overleftrightarrow{1} - \vec{r}\vec{r}\right)$, we have

$$\int (d\vec{r})\,\rho(\vec{r})\vec{r}\vec{r} = \underbrace{\rho_0\int (d\vec{r})\,\vec{r}\vec{r}}_{ball} + \underbrace{\rho_0\int (d\vec{r})\,\vec{r}\vec{r}}_{ellipsoid} = \overleftrightarrow{B} + \overleftrightarrow{E}$$

with

$$\overleftrightarrow{B} = \rho_0\int_{r<R}(d\vec{r})\vec{r}\vec{r} = \rho_0\frac{1}{3}\overleftrightarrow{1}\int_{r<R}(d\vec{r})r^2 = \rho_0\frac{4\pi}{3}\frac{R^5}{5}\overleftrightarrow{1}$$

and

$$\overleftrightarrow{E} = \rho_0\int (d\vec{r})\,\vec{r}\vec{r}\eta\left(1 - \frac{x^2+y^2}{4R^2} - \frac{z^2}{R^2}\right) \qquad [x = 2x', y = 2y',\ \ z = z']$$

$$= \rho_0 4\int_{r'<R}(d\vec{r}\,')\left(4x'^2\vec{e}_x\vec{e}_x + 4y'^2\vec{e}_y\vec{e}_y + z'^2\vec{e}_z\vec{e}_z\right)$$

$$= \rho_0\frac{4}{3}\int_{r'<R}(d\vec{r}\,')\,r'^2\left(4\vec{e}_x\vec{e}_x + 4\vec{e}_y\vec{e}_y + \vec{e}_z\vec{e}_z\right)$$

$$= \rho_0\frac{4\pi}{3}\frac{4R^5}{5}\left(4\overleftrightarrow{1} - 3\vec{e}_z\vec{e}_z\right)$$

[$x = 2x', y = 2y',\ \ z = z'$. Terms like $xy\vec{e}_x\vec{e}_y$ are odd and do not contribute.]

Together,

$$\overleftrightarrow{B} + \overleftrightarrow{E} = \rho_0\frac{4\pi}{3}\frac{R^5}{5}\left(17\overleftrightarrow{1} - 12\vec{e}_z\vec{e}_z\right) = \frac{MR^2}{25}\left(17\overleftrightarrow{1} - 12\vec{e}_z\vec{e}_z\right)$$

so that

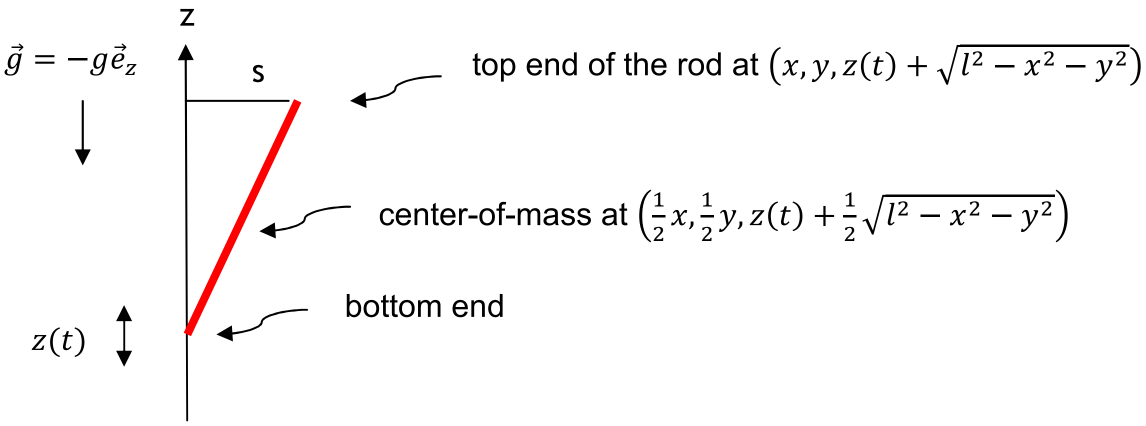
$$\overleftrightarrow{I} = \overleftrightarrow{\mathbb{1}} tr\{\overleftrightarrow{B} + \overleftrightarrow{E}\} - (\overleftrightarrow{B} + \overleftrightarrow{E}) = \frac{MR^2}{25}\left[29\overleftrightarrow{\mathbb{1}} - (17\overleftrightarrow{\mathbb{1}} - 12\vec{e}_z\vec{e}_z)\right] = \frac{MR^2}{25}(22\overleftrightarrow{\mathbb{1}} + 12\vec{e}_z\vec{e}_z)$$

Question 4(b)

The angular momentum is

$$\vec{L} = \overleftrightarrow{I}\cdot\vec{\omega} = \frac{MR^2}{25}\Bigg(22\vec{\omega} + 12\vec{e}_z \underbrace{\vec{e}_z\cdot\vec{\omega}}_{\omega\cos\theta}\Bigg) = \frac{MR^2}{25}\omega(22\vec{e}_x\sin\theta + 34\vec{e}_z\cos\theta)$$

Question 5(a)



A mass element of the rod at $\left(ux, uy, z(t) + u\sqrt{l^2 - x^2 - y^2}\right)$, $0 \le u \le 1$, has velocity $\left(u\dot{x}, u\dot{y}, \dot{z} - \frac{u(x\dot{x}+y\dot{y})}{\sqrt{l^2-x^2-y^2}}\right)$, so that the kinetic energy is

$$\int_0^1 du \frac{1}{2}M\left[u^2\dot{x}^2 + u^2\dot{y}^2 + \dot{z}^2 - \frac{2u\dot{z}(x\dot{x}+y\dot{y})}{\sqrt{l^2-x^2-y^2}} + \frac{u^2(x\dot{x}+y\dot{y})^2}{l^2-x^2-y^2}\right]$$

$$= \frac{M}{6}\left[\dot{x}^2 + \dot{y}^2 + 3\dot{z}^2 - \frac{3\dot{z}(x\dot{x}+y\dot{y})}{\sqrt{l^2-x^2-y^2}} + \frac{(x\dot{x}+y\dot{y})^2}{l^2-x^2-y^2}\right]$$

$$= \frac{M}{6}\left[\dot{x}^2 + \dot{y}^2 - \frac{3\dot{z}(x\dot{x}+y\dot{y})}{l}\right] + \cdots,$$

where the ellipsis stands for the terms of higher than quadratic order in $x, \dot{x}, y, \dot{y}$ and for the $\dot{z}^2$ term which is independent of these coordinates.

We combine this with the potential energy

$$Mg\left[z(t) + \frac{1}{2}\sqrt{l^2 - x^2 - y^2}\right] = -\frac{1}{4}Mg\frac{x^2+y^2}{l} + \cdots$$

to the Lagrange function for motion restricted to small values of $s = \sqrt{x^2+y^2}$, that is $s \ll l$,

$$L = \frac{M}{6}(\dot{x}^2 + \dot{y}^2) - \frac{M}{2l}(x\dot{x} + y\dot{y}) + \frac{Mg}{4l}(x^2 + y^2)$$

The Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} : \frac{d}{dt}\left(\frac{1}{3}M\dot{x} - \frac{M}{2l}\dot{z}x\right) = -\frac{M}{2l}\dot{z}\dot{x} + \frac{Mg}{2l}x$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y} : \frac{d}{dt}\left(\frac{1}{3}M\dot{y} - \frac{M}{2l}\dot{z}y\right) = -\frac{M}{2l}\dot{z}\dot{y} + \frac{Mg}{2l}y$$

which is the same equation twice, so that it is enough to look at the equation for $x(t)$, namely

$$\ddot{x} = \frac{3}{2l}(g + \ddot{z})x = \frac{3g}{2l}(1 \pm \lambda)x$$

Question 5(b)

We need a restoring force for one half-period, so that $\lambda > 1$ is a first condition. Then $\ddot{x} = \omega^2 + x$ for half a period, and $\ddot{x} = -\omega^2 - x$ for the other half period, with

$$\omega_\pm^2 = \frac{3g}{2l}(\lambda \pm 1).$$

The mappings

$$\begin{pmatrix} x(t) \\ T\dot{x}(t) \end{pmatrix} \to \begin{pmatrix} x\left(t + \frac{1}{2}T\right) \\ T\dot{x}\left(t + \frac{1}{2}T\right) \end{pmatrix} \to \begin{pmatrix} x(t+T) \\ T\dot{x}(t+T) \end{pmatrix}$$

are accomplished by the 2×2 matrices

$$M_+ = \begin{pmatrix} \cosh\left(\omega + \frac{T}{2}\right) & \frac{1}{\omega + T}\sinh\left(\omega + \frac{T}{2}\right) \\ \omega + T\sinh\left(\omega + \frac{T}{2}\right) & \cosh\left(\omega + \frac{T}{2}\right) \end{pmatrix}$$

and

$$M_- = \begin{pmatrix} \cos\left(\omega - \frac{T}{2}\right) & \frac{1}{\omega - T}\sin\left(\omega - \frac{T}{2}\right) \\ -\omega - T\sin\left(\omega - \frac{T}{2}\right) & \cos\left(\omega - \frac{T}{2}\right) \end{pmatrix}$$

The rod stays upright of the eigenvalues of M_+M_- (or M_-M_+) do not have absolute values that exceed unity. Upon denoting these eigenvalues by M_1 and M_2 we have

$$M_1M_2 = \det\{M_+M_-\} = \det\{M_+\}\det\{M_-\} = 1$$

and

$$\mu_1 + \mu_2 = tr\{M_+M_-\} = 2\cosh\left(\frac{\omega + T}{2}\right)\cos\left(\frac{\omega - T}{2}\right).$$

So we have 2 real eigenvalues $\mu_1 = \pm e^{\alpha}$, $\mu_2 = \pm e^{-\alpha}$ if $\cosh\alpha = \left|\cosh\left(\frac{\omega+T}{2}\right)\cos\left(\frac{\omega-T}{2}\right)\right| > 1$, and one of these values is too big, or we have a pair of complex phase factors, $\mu_1 = e^{i\alpha}$, $\mu_2 = e^{-i\alpha}$ if $|\cos\alpha| \left|\cosh\left(\frac{\omega+T}{2}\right)\cos\left(\frac{\omega-T}{2}\right)\right| \leq 1$.

The conditions to be met by λ and T is, therefore $-1 \leq \cosh\left(\frac{\omega+T}{2}\right)\cos\left(\frac{\omega-T}{2}\right) \leq 1$ with $\omega_{\pm} = \sqrt{\frac{3g}{2l}(\lambda \pm 1)}$, in addition to $\lambda > 1$.

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