PC2132

NATIONAL UNIVERSITY OF SINGAPORE

PC2132 – Classical Mechanics
(Lecturer: B.-G. Englert)

(Semester I: AY2011/12)

Exam, 23 November 2011

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE** questions and comprises **THREE** printed pages.
2. Answer **ALL FIVE** questions for a total of 100 marks.
3. Show all your work in the answer book.
4. For each question, **clearly** indicate what constitutes your final answer.
5. Lecture notes for PC2132 and personal notes directly related to the module may be consulted during the test, **but no other printed or written material**.
6. The use of **electronic equipment** of any kind **is not permitted**.

1. **Conservative force (10 marks)**
   Is the centrifugal force $\vec{F}(\vec{r}) = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ conservative? If yes, state its potential energy.

2. **One-dimensional periodic motion (20=6+9+5 marks)**
   A point mass m moves along the x axis under the influence of the force

$$F(x) = -a(x^2 - x_0^2)\,,$$

   where a and x_0 are positive constants.

   (a) The force vanishes for $x = \pm x_0$. Which of them is the location of a stable equilibrium?

   (b) What is the period of small-amplitude oscillations about the stable-equilibrium position?

   (c) Without any calculation: Is the period longer or shorter for oscillations with amplitudes that are not small? Explain.

3. **Lagrange function, Hamilton function (20=10+10 marks)**
   The dynamics of point particle (mass m, position $\vec{r}(t)$, velocity $\vec{v}(t)$) is described by the Lagrange function

$$L = \frac{m}{2}\vec{v}^2 - V(\vec{r}) + \vec{v}\cdot\vec{\nabla}u(\vec{r})\,,$$

   where $V(\vec{r})$ is the potential energy and $u(\vec{r})$ is some given function of position $\vec{r}$.

   (a) Derive the Newton's equation of motion for the particle and so show that the same physical system is described, irrespective of which $u(\vec{r})$ is chosen.

   (b) Find the corresponding Hamilton function. Does it depend on the choice of $u(\vec{r})$?

4. Moments of inertia (20=15+5 marks)

A rigid body of mass M has the shape of an oblate ellipsoid whose surface is given by

$$x^2 + y^2 + 4z^2 = 4R^2 \quad \text{with} \quad R > 0\,.$$

The body has a homogeneous mass density ρ_0, except for a ball-shaped core of radius R that has density $2\rho_0$.

(a) Find the inertia dyadic in terms of M and R.

(b) The body rotates with angular velocity $\vec{\omega} = \omega(\vec{e}_x \sin\theta + \vec{e}_z \cos\theta)$ about an axis through its center. What is the angular momentum?

5. Upside-down pendulum (30=15+15 marks)

A thin rigid homogeneous rod of length ℓ and mass m is standing upright, but is slightly off the exact vertical position and tends to fall over because the gravitational acceleration g is pulling it down. The bottom end of the rod is periodically moved vertically up and down along the z axis, such that its acceleration switches between λg and $-\lambda g$ at regular instants separated by the half-period $T/2$, whereby λ is a positive constant parameter.

(a) With the top of the rod at distance $s(t) = \sqrt{x(t)^2 + y(t)^2}$ from the z axis, state the equations of motion for $x(t)$ and $y(t)$ when $s \ll \ell$.

(b) Which condition must be met by λ and T, so that the rod stays upright and does not fall over?

End of Paper