NATIONAL UNIVERSITY OF SINGAPORE

PC2134 Mathematical Methods in Physics I

(Special Term II: AY 2015 – 16)

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. Please write your student number only. Do not write your name.
2. This assessment paper contains **4** questions and comprises **4** printed pages.
3. Students are required to answer **ALL** questions. The answers are to be written on the answer books.
4. Students should write the answers for each question on a new page.
5. This is a CLOSED BOOK examination.
6. Programmable calculators are **NOT** allowed.
7. All questions carry equal marks. The total mark is 60.

Question 1

(a) Find the equations of the normal line and tangent plane to the surface $f(x,y,z) = 0$, with

$$f(x,y,z) = -\cos(xyz) + 2x - 2,$$

at the point $(1, \pi/2, 1)$.

[5]

(b) Consider the vector field

$$\mathbf{A}(x,y,z) = \sinh(x-z)\,\mathbf{e}_x + 2y\mathbf{e}_y - xyz\mathbf{e}_z.$$

Calculate $\nabla \cdot \mathbf{A}$ and $\nabla \times \mathbf{A}$.

[5]

(c) Let

$$\mathbf{A}(x,y,z) = A_x(x,y,z)\mathbf{e}_x + A_y(x,y,z)\mathbf{e}_y + A_z(x,y,z)\mathbf{e}_z$$

and

$$\mathbf{B}(x,y,z) = B_x(x,y,z)\mathbf{e}_x + B_y(x,y,z)\mathbf{e}_y + B_z(x,y,z)\mathbf{e}_z$$

be vector fields. Show that

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B}).$$

[5]

Question 2

(a) Consider a periodic function $f = f(x)$ with period L. For $0 \le x \le L$, the Fourier series representation of $f(x)$ is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{L},$$

where

$$a_n = \frac{2}{L}\int_0^L \cos\frac{2n\pi x}{L} f(x)dx, \qquad b_n = \frac{2}{L}\int_0^L \sin\frac{2n\pi x}{L} f(x)dx.$$

Show that

$$\frac{1}{L}\int_0^L [f(x)]^2 dx = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2.$$

[5]

(b) The Fourier transform of a function $f = f(x)$,

$$F[f(x)] \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\exp(-ikx)\,dx = g(k).$$

i. Consider the Dirac delta function $\delta(x-a)$, where a is some real constant. Find $F[\delta(x-a)]$, and hence derive the Fourier transform representation of $\delta(x-a)$.

[2]

ii. Hence, show that

$$\int_{-\infty}^{\infty} |g(k)|^2 dk = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

[3]

(c) Suppose

$$f(x) = \begin{cases} x & \text{for} \quad -3 \le x \le 3, \\ 0 & \text{for} \quad |x| > 3. \end{cases}$$

Find the corresponding $g(k)$ and calculate

$$\int_{-\infty}^{\infty} |g(k)|^2 dk.$$

[5]

Question 3

The Laplace transform of a differentiable function $f = f(x)$,

$$L[f(x)] \equiv \int_0^{\infty} f(x) \exp(-kx)\, dx = g(k).$$

In the following, you may use the fact that operator L is linear without proof.

(a) Show that

$$L[f(ax)] = \frac{1}{a} g\left(\frac{k}{a}\right),$$

where a is some real positive constant.

[2]

(b) Show that

$$L[f'(x)] = kL[f(x)] - f(0),$$
$$L[f''(x)] = k^2 L[f(x)] - kf(0) - f'(0).$$

[4]

(c) Calculate $L[\exp(x)]$ and hence $L[\cosh(3x)]$.

[4]

(d) Solve the following differential equation using Laplace transform method.

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = \cosh(3x), \qquad y(0) = 0, \qquad y'(0) = 0.$$

[5]

You will find the following results useful:

$$L^{-1}[g_1(k) g_2(k)] = f_1 * f_2 = \int_0^x f_1(x-y) f_2(y) dy,$$

$$\int_0^x \exp[-a(x-y)] \cosh(by)\, dy = -\frac{a\exp(-ax) - a\cosh(bx) + b\sinh(bx)}{a^2 - b^2},$$

$$\int_0^x \exp[-a(x-y)] \cosh(ay)\, dy = \frac{\exp(-ax)\,[\exp(2ax) + 2ax - 1]}{4a}.$$

Question 4

Consider the *Euler's equation*

$$x^2\frac{d^2y}{dx^2}+ax\frac{dy}{dx}+by=0,$$

where a and b are some constants.

(a) Find the general solution of the equation.

[5]

(b) Explain briefly how $x=0$ is a regular singular point of the above equation. And, show that $x=\infty$ is also a regular singular point.

[5]

(c) Find the general solution of the equation

$$(x-1)^2\frac{d^2y}{dx^2}+a(x-1)\frac{dy}{dx}+by=0,$$

as a power series about the regular point $x=0$. Here, a and b are some constants. You only need to write down the first four nonzero terms of the series.

[5]

– End of Paper –

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