NATIONAL UNIVERSITY OF SINGAPORE

PC2134 MATHEMATICAL METHODS IN PHYSICS I

(Semester I: AY 2016-17)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name.**
2. This examination paper contains **FOUR (4)** questions and comprises **THREE (3)** printed pages.
3. Students are required to answer **ALL** questions.
4. Students should write the answers for each question on a new page.
5. This is a **CLOSED** book examination. You are allowed however to bring in **ONE** A4-sized (both sides) sheet of notes.
6. Specific permitted devices: non-programmable calculators.

Question 1 **[25=5+10+10]**

(a) Consider a general transformation between two different sets of variables $(x_1, x_2) \leftrightarrow (y_1, y_2)$:

$$\begin{cases} x_1 = x_1(y_1, y_2) \\ x_2 = x_2(y_1, y_2) \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} y_1 = y_1(x_1, x_2) \\ y_2 = y_2(x_1, x_2) \end{cases},$$

where all functions involved are differentiable. Prove the reciprocal rule of the Jacobian:

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \left[\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right]^{-1}.$$

(b) Evaluate the following integral

$$I = \int_0^\infty \int_0^\infty \frac{x^2 + y^2}{1 + (x^2 - y^2)^2} \exp(-2xy)\, \mathrm{d}x\, \mathrm{d}y\,,$$

by making a change of variables

$$u = x^2 - y^2\,, \qquad v = 2xy\,.$$

(c) Evaluate the following surface integral

$$\iint_{\mathcal{S}} \Phi(x, y, z)\, \mathrm{d}\mathbf{S}\,, \qquad \Phi(x, y, z) = \frac{3}{8}\, xyz\,,$$

where $\mathcal{S}$ is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Question 2 **[25=10+15]**

The general form of the second-order linear ordinary differential equation is given as follows:

$$\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + P(x)\, \frac{\mathrm{d}y(x)}{\mathrm{d}x} + Q(x)\, y(x) = f(x)\,.$$

The general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)\,,$$

where $y_{1,2}(x)$ are two linearly independent solutions of the homogeneous equation corresponding to $f(x) = 0$, $y_p(x)$ is the solution of the inhomogeneous equations and $c_{1,2}$ are two arbitrary constants.

(a) Given that $y_1(x)$ satisfies the homogeneous second-order linear ordinary differential equation, show that

$$y_2(x) = y_1(x) \int^x \frac{\exp\left[-\int^u P(v)\, \mathrm{d}v\right]}{[y_1(u)]^2}\, \mathrm{d}u\,,$$

also satisfies the differential equation.

(b) Consider the following differential equation,

$$x^2\, \frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} - x\, \frac{\mathrm{d}y(x)}{\mathrm{d}x} + y(x) = (\ln x)^2\,.$$

Given that $y_1(x) = x$ is one of the solutions of the homogeneous equation, find the general solution of the inhomogeneous equation.

Question 3 **[25=10+15]**

The complex Fourier series of a function $f(x)$ of periodicity $2L$ is given as follows:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\mathrm{i}\,\frac{n\pi x}{L}\right),$$

where c_n is known as the complex Fourier coefficient.

(a) Derive the expression for the complex Fourier coefficient and the Parseval theorem for the complex Fourier series.

(b) Consider a full-wave rectifier that accepts an input signal $\sin x$, $-\pi \leq x \leq \pi$, and produces as output of the signal: $V(x) = |\sin x|$ for $-\pi \leq x \leq \pi$. Estimate the ratio of the DC output with respect to the incoming total input energy.

Question 4 **[25=5+10+10]**

The Fourier transform pair $f(\mathbf{r})$ and $\tilde{f}(\mathbf{k})$ in three dimensions are defined as follows:

$$\mathcal{F}\{f(\mathbf{r})\} = \left(\frac{1}{2\pi}\right)^{3/2} \iiint \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r})\,\mathrm{d}^3\mathbf{r} \quad \Leftrightarrow \quad \mathcal{F}^{-1}\left\{\tilde{f}(\mathbf{k})\right\} = \left(\frac{1}{2\pi}\right)^{3/2} \iiint \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \tilde{f}(\mathbf{k})\,\mathrm{d}^3\mathbf{k}\,,$$

where the integrations are over the entire (x, y, z) or (k_x, k_y, k_z) space.

(a) Given that $\mathbf{A}(\mathbf{r})$ is an arbitrary vector function and $\Phi(\mathbf{r})$ is an arbitrary scalar function, prove that

$$\boldsymbol{\nabla} \cdot (\Phi\mathbf{A}) = \Phi\,(\boldsymbol{\nabla} \cdot \mathbf{A}) + (\boldsymbol{\nabla}\Phi) \cdot \mathbf{A}\,.$$

(b) Show that the Fourier transform of the Laplacian of an arbitrary scalar function $\Phi(\mathbf{r})$ is given as follow:

$$\mathcal{F}\left\{\nabla^2\Phi(\mathbf{r})\right\} = -k^2\,\tilde{\Phi}(\mathbf{k})\,,$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$ and $\tilde{\Phi}(\mathbf{k})$ is the Fourier transform of $\Phi(\mathbf{r})$. List down any assumption made on the scalar function $\Phi(\mathbf{r})$ in your derivation.

(c) In Newtonian gravity, the Poisson equation relates gravitational potential $\Phi(\mathbf{r})$ to the volume mass density $\rho(\mathbf{r})$ as follows:

$$\nabla^2\Phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r})\,,$$

where G is the gravitational constant. For a given volume mass density $\rho(\mathbf{r})$, use the Fourier transform and the Fourier convolution theorem to show that $\Phi(\mathbf{r})$ can be written in the following form:

$$\Phi(\mathbf{r}) = -G \iiint \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'\,.$$

Useful information:

$$\mathcal{F}\left\{\frac{\mathrm{e}^{-\alpha r}}{r}\right\} = \left(\frac{1}{2\pi}\right)^{3/2} \frac{4\pi}{k^2 + \alpha^2}$$

$$f(\mathbf{r}) * g(\mathbf{r}) = \iiint f(\mathbf{r}')\,g(\mathbf{r} - \mathbf{r}')\,\mathrm{d}^3\mathbf{r}'$$

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END OF PAPER