NATIONAL UNIVERSITY OF SINGAPORE

PC2134 MATHEMATICAL METHODS IN PHYSICS I

(Semester I: AY 2017-18)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name.**
2. This examination paper contains **FOUR (4)** questions and comprises **THREE (3)** printed pages.
3. Students are required to answer **ALL** questions.
4. Students should write the answers for each question on a new page.
5. This is a **CLOSED** book examination. You are allowed however to bring in **ONE** A4-sized help sheet.
6. Specific permitted devices: non-programmable calculators.

Question 1 **[25=15+10]**

In classical thermodynamics, the exact differential of internal energy $E(S,V)$, for a system of gas with fixed number of particles, satisfies the following fundamental relation:

$$\mathrm{d}E = T\,\mathrm{d}S - P\,\mathrm{d}V\,,$$

where S is the entropy, and P, V and T are the pressure, volume and temperature of the system respectively.

(a) Construct a function $G(T,P)$, known as Gibbs energy, via a Legendre transformation such that

$$S = -\left(\frac{\partial G}{\partial T}\right)_P\,, \qquad \text{and} \qquad V = \left(\frac{\partial G}{\partial P}\right)_T\,.$$

Hence, show that

$$\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P\,.$$

(b) A certain gas system is found to have a Gibbs energy given by

$$G(T,P) = RT\ln\left[\frac{aP}{(RT)^{5/2}}\right]$$

where a and R are constants. Find the specific heat capacity at constant pressure

$$C_P \equiv T\left(\frac{\partial S}{\partial T}\right)_P\,,$$

of this gas system.

Question 2 **[25=10+15]**

In non-relativistic quantum mechanics, the position wavefunction $\psi(x)$ of a quantum particle subjected to a time-independent potential energy function $V(x)$ in one-dimensional world is to satisfy the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + V(x)\,\psi(x) = E\,\psi(x)\,,$$

where $\hbar$ is the reduced Planck's constant, $\hbar \equiv h/2\pi$ and E is the energy associated with the position wavefunction $\psi(x)$.

(a) The momentum wavefunction $\Phi(p)$ and position wavefunction $\psi(x)$ are related by Fourier transform below:

$$\Phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\exp\left(-\frac{ipx}{\hbar}\right)\psi(x)\,\mathrm{d}x\,.$$

Show that the momentum wavefunction satisfies the following equation:

$$\frac{p^2}{2m}\,\Phi(p) + \int_{-\infty}^{\infty}\tilde{V}(p-p')\Phi(p')\,\mathrm{d}p' = E\,\Phi(p)\,.$$

What is the expression for $\tilde{V}(p,p')$?

(b) If $V(x) = -\alpha\,\delta(x)$ where $\alpha > 0$, show that the momentum wavefunction is given by

$$\Phi(p) = \frac{m\alpha}{\pi\hbar}\frac{C}{p^2 + 2m|E|}\,.$$

Identify the expression for C, in terms of $\Phi(p)$, and hence determine the value of E.

Question 3 **[25=10+15]**

In electromagnetism, magnetostatic vector potential $\mathbf{A}(\mathbf{r})$ and volume current density $\mathbf{J}(\mathbf{r})$ are related by

$$\boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r})] = \mu_0 \mathbf{J}(\mathbf{r}) \, ,$$

where μ_0 is the permeability of vacuum. A current distribution produces the following magnetostatic vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{A_0 \sin\theta}{r} \exp(-\lambda r) \; \hat{\mathbf{e}}_\phi \, ,$$

where A_0 and λ are constants.

(a) Find the volume current density of this distribution.

(b) The magnetic dipole moment m is defined by

$$\mathrm{m} = \frac{1}{2} \iiint \mathrm{r} \times \mathbf{J}(\mathbf{r}) \, \mathrm{d}V \, ,$$

where the integration is carried out in the entire region of the current distribution. Find the magnetic dipole moment associated with this current distribution.

Question 4 **[25=10+15]**

The Laplace transform of the function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} \equiv \overline{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t) \, \mathrm{d}t \, ,$$

provided that the integral exists.

(a) Prove the Laplace convolution theorem:

$$\mathcal{L}\{f(t) * g(t)\} \equiv \int_0^\infty \int_0^t \mathrm{e}^{-st} f(u) \, g(t-u) \, \mathrm{d}u \, \mathrm{d}t = \overline{f}(s) \, \overline{g}(s) \, .$$

(b) In mechanics, the motion of a driven damped harmonic oscillator may be described by the following second order differential equation:

$$m \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + b \frac{\mathrm{d}x(t)}{\mathrm{d}t} + kx(x) = F(t) \, , \qquad x(0) = \left. \frac{\mathrm{d}x(t)}{\mathrm{d}t} \right|_{t=0} = 0 \, , \qquad t \geq 0 \, ,$$

where m is the mass of the oscillating particle, k is the spring constant, b is the proportional constant for the damping force and $F(t)$ is the driving force.

(i) Show that the solution can be written in the following form:

$$x(t) = \frac{1}{m\omega} \int_0^t \mathrm{e}^{-\alpha(t-\tau)} f(\tau) \sin[\omega(t-\tau)] \, \mathrm{d}\tau \, .$$

What are the expressions for α and ω?

(ii) If $b = 0$ and $F(t) = F_0 \, H(t - t_0)$ where $H(t - t_0)$ is the Heaviside step function and F_0 is a constant, solve for $x(t)$ using Laplace transform.

KHCM

END OF PAPER