NATIONAL UNIVERSITY OF SINGAPORE

PC2134 MATHEMATICAL METHODS IN PHYSICS I

(Semester I: AY 2017-18)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name.**
2. This examination paper contains **FOUR (4)** questions and comprises **THREE (3)** printed pages.
3. Students are required to answer **ALL** questions.
4. Students should write the answers for each question on a new page.
5. This is a **CLOSED** book examination. You are allowed however to bring in **ONE** A4-sized help sheet.
6. Specific permitted devices: non-programmable calculators.

Question 1 **[25=15+10]**

In classical thermodynamics, the exact differential of internal energy $E(S,V)$, for a system of gas with fixed number of particles, satisfies the following fundamental relation:

$$\mathrm{d}E = T\,\mathrm{d}S - P\,\mathrm{d}V\,,$$

where S is the entropy, and P, V and T are the pressure, volume and temperature of the system respectively.

(a) Construct a function $G(T,P)$, known as Gibbs energy, via a Legendre transformation such that

$$S = -\left(\frac{\partial G}{\partial T}\right)_P\,, \qquad \text{and} \qquad V = \left(\frac{\partial G}{\partial P}\right)_T\,.$$

Hence, show that

$$\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P\,.$$

Solution:

$$G(T,P) \equiv E + PV - TS \quad \Rightarrow \quad \mathrm{d}G = -S\,\mathrm{d}T + V\,\mathrm{d}P \qquad \blacksquare$$

$$\left[\frac{\partial}{\partial P}\left(\frac{\partial G}{\partial T}\right)_P\right]_T = \left[\frac{\partial}{\partial T}\left(\frac{\partial G}{\partial P}\right)_T\right]_P \quad \Rightarrow \quad \left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P \qquad \blacksquare$$

(b) A certain gas system is found to have a Gibbs energy given by

$$G(T,P) = RT\ln\left[\frac{aP}{(RT)^{5/2}}\right]$$

where a and R are constants. Find the specific heat capacity at constant pressure

$$C_P \equiv T\left(\frac{\partial S}{\partial T}\right)_P\,,$$

of this gas system.

Solution:

$$S(T,P) = -\left(\frac{\partial G}{\partial T}\right)_P = \frac{5}{2}R - R\,\ln\left[\frac{aP}{(RT)^{5/2}}\right]$$

$$C_P = T\left(\frac{\partial S}{\partial T}\right)_P = \frac{5}{2}R \qquad \blacksquare$$

Question 2 **[25=10+15]**

In non-relativistic quantum mechanics, the position wavefunction $\psi(x)$ of a quantum particle subjected to a time-independent potential energy function $V(x)$ in one-dimensional world is to satisfy the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2}+V(x)\,\psi(x)=E\,\psi(x)\,,$$

where $\hbar$ is the reduced Planck's constant, $\hbar\equiv h/2\pi$ and E is the energy associated with the position wavefunction $\psi(x)$.

(a) The momentum wavefunction $\Phi(p)$ and position wavefunction $\psi(x)$ are related by Fourier transform below:

$$\Phi(p)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\exp\left(-\frac{\mathrm{i}px}{\hbar}\right)\psi(x)\,\mathrm{d}x\,.$$

Show that the momentum wavefunction satisfies the following equation:

$$\frac{p^2}{2m}\,\Phi(p)+\int_{-\infty}^{\infty}\tilde{V}(p-p')\Phi(p')\,\mathrm{d}p'=E\,\Phi(p)\,.$$

What is the expression for $\tilde{V}(p-p')$?

(b) If $V(x)=-\alpha\,\delta(x)$ where $\alpha>0$, show that the momentum wavefunction is given by

$$\Phi(p)=\frac{m\alpha}{\pi\hbar}\,\frac{C}{p^2+2m|E|}\,.$$

Identify the expression for C, in terms of $\Phi(p)$, and hence determine the value of E.

Solution:

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2}+V(x)\,\psi(x)=E\,\psi(x)\quad\Rightarrow\quad\frac{p^2}{2m}\,\Phi(p)+\frac{1}{\sqrt{2\pi\hbar}}\tilde{V}(p)*\Phi(p)=E\,\Phi(p)$$

$$\Rightarrow\quad\tilde{V}(p-p')=\frac{1}{2\pi\hbar}\int_{\infty}^{\infty}\exp\left[-\frac{\mathrm{i}\,(p-p')\,x}{\hbar}\right]V(x)\,\mathrm{d}x\qquad\blacksquare$$

$$\tilde{V}(p-p')=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\exp\left[-\frac{\mathrm{i}\,(p-p')\,x}{\hbar}\right][-\alpha\,\delta(x)]\,\mathrm{d}x=-\frac{\alpha}{2\pi\hbar}$$

$$\frac{p^2}{2m}-\frac{\alpha}{2\pi\hbar}\int_{-\infty}^{\infty}\Phi(p')\,\mathrm{d}p'=E\,\Phi(p)\quad\Rightarrow\quad C=\int_{-\infty}^{\infty}\Phi(p')\,\mathrm{d}p'\qquad\blacksquare$$

$$\frac{p^2}{2m}-\frac{\alpha}{2\pi\hbar}C=E\,\Phi(p)\quad\Rightarrow\quad\Phi(p)=\frac{m\alpha}{\pi\hbar}\,\frac{C}{p^2-2mE}\qquad\blacksquare$$

$$C=\frac{m\alpha}{\pi\hbar}\int_{-\infty}^{\infty}\frac{C}{p'^2-2mE}\,\mathrm{d}p'\quad\Rightarrow\quad\frac{m\alpha}{\pi\hbar}\int_{-\infty}^{\infty}\frac{1}{p'^2+\left(\sqrt{-2mE}\right)^2}\,\mathrm{d}p'=1$$

$$\Rightarrow\quad\frac{m\alpha}{\pi\hbar}\,\frac{\pi}{\sqrt{-2mE}}=1\quad\Rightarrow\quad E=-\frac{m\alpha^2}{2\hbar^2}\qquad\blacksquare$$

Question 3 **[25=10+15]**

In electromagnetism, magnetostatic vector potential $\mathbf{A}(\mathbf{r})$ and volume current density $\mathbf{J}(\mathbf{r})$ are related by

$$\boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r})] = \mu_0 \mathbf{J}(\mathbf{r}) \,,$$

where μ_0 is the permeability of vacuum. A current distribution produces the following magnetostatic vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{A_0 \sin\theta}{r} \exp(-\lambda r) \; \hat{\mathbf{e}}_\phi \,,$$

where A_0 and λ are constants.

(a) Find the volume current density of this distribution.

(b) The magnetic dipole moment $\mathbf{m}$ is defined by

$$\mathbf{m} = \frac{1}{2} \iiint \mathbf{r} \times \mathbf{J}(\mathbf{r}) \, \mathrm{d}V \,,$$

where the integration is carried out in the entire region of the current distribution. Find the magnetic dipole moment associated with this current distribution.

Solution:

$$\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}) = \frac{\mu_0 A_0}{4\pi} \exp(-\lambda r) \left(\frac{2\cos\theta}{r^2} \, \hat{\mathbf{e}}_r + \frac{\lambda \sin\theta}{r} \, \hat{\mathbf{e}}_\theta \right]$$

$$\mathbf{J}(\mathbf{r}) = \frac{A_0}{4\pi} \sin\theta \, \mathrm{e}^{-\lambda r} \left(\frac{2}{r^3} - \frac{\lambda^2}{r} \right) \hat{\mathbf{e}}_\phi \qquad \blacksquare$$

$$\mathbf{m} = -\frac{A_0}{8\pi} \iiint r \exp(-\lambda r) \left(\frac{2}{r^3} - \frac{\lambda^2}{r} \right) \sin\theta \, \hat{\mathbf{e}}_\theta \, \mathrm{d}V = \mathbf{0} \qquad \blacksquare$$

Question 4 **[25=10+15]**

The Laplace transform of the function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} \equiv \overline{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t\,,$$

provided that the integral exists.

(a) Prove the Laplace convolution theorem:

$$\mathcal{L}\{f(t) * g(t)\} \equiv \int_0^\infty \int_0^t \mathrm{e}^{-st} f(u)\, g(t-u)\,\mathrm{d}u\,\mathrm{d}t = \overline{f}(s)\,\overline{g}(s)\,.$$

(b) In mechanics, the motion of a driven damped harmonic oscillator may be described by the following second order differential equation:

$$m\,\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + b\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} + kx(t) = F(t)\,, \qquad x(0) = \left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=0} = 0\,, \qquad t \geq 0\,,$$

where m is the mass of the oscillating particle, k is the spring constant, b is the proportional constant for the damping force and $F(t)$ is the driving force.

(i) Show that the solution can be written in the following form:

$$x(t) = \frac{1}{m\omega} \int_0^t \mathrm{e}^{-\alpha(t-\tau)}\, F(\tau)\, \sin\left[\omega\,(t-\tau)\right]\,\mathrm{d}\tau\,.$$

What are the expressions for α and ω?

(ii) If $b = 0$ and $F(t) = F_0\, H(t - t_0)$ where $H(t - t_0)$ is the Heaviside step function and F_0 is a constant, solve for $x(t)$ using Laplace transform.

Solution:

$$m\,\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + b\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} + kx(t) = F(t) \quad\Rightarrow\quad \overline{x}(s) = \frac{\overline{F}(s)}{ms^2 + bs + k} \equiv \frac{\overline{F}(s)}{m\left[(s-\alpha)^2 + \omega^2\right]}$$

$$\Rightarrow \quad \alpha = \frac{b}{2m}\,, \qquad \omega^2 = \frac{k}{m} - \left(\frac{b}{2m}\right)^2 \qquad \blacksquare$$

$$x(t) = \mathcal{L}^{-1}\left\{\overline{F}(s)\,\overline{G}(s)\right\}\,, \qquad G(t) \equiv \frac{1}{m\omega}\,\mathrm{e}^{-\alpha t}\,\sin(\omega t)$$

$$x(t) = F(t) * G(t) = \frac{1}{m\omega}\int_0^t \mathrm{e}^{-\alpha(t-\tau)}\, F(\tau)\,\sin\left[\omega\,(t-\tau)\right]\,\mathrm{d}\tau \qquad \blacksquare$$

$$x(t) = \frac{1}{m\omega}\int_0^t F_0\, H(\tau - t_0)\,\sin\left[\omega\,(t-\tau)\right]\,\mathrm{d}\tau = \frac{F_0}{m\omega}\int_{t_0}^t \sin\left[\omega\,(t-\tau)\right]\,\mathrm{d}\tau = \frac{F_0}{m\omega^2}\Big\{1 - \cos\left[\omega\,(t-t_0)\right]\Big\}$$

$$\Rightarrow \quad x(t) = \frac{F_0}{m\omega^2}\Big\{1 - \cos\left[\omega\,(t-t_0)\right]\Big\}\, H\,(t - t_0) \qquad \blacksquare$$

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END OF PAPER