NATIONAL UNIVERSITY OF SINGAPORE

PC2134 Mathematical Methods in Physics I

(Special Term II: AY 2017 – 18)

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. Please write your student number only. Do not write your name.
2. This assessment paper contains **4** questions and comprises **3** printed pages.
3. Students are required to answer **ALL** questions. The answers are to be written on the answer books.
4. Students should write the answers for each question on a new page.
5. This is a CLOSED BOOK examination.
6. Programmable calculators are **NOT** allowed.
7. All questions carry equal marks. The total mark is 60.

Question 1 Fourier series and Fourier transforms

(a) Derive a Fourier cosine series for the function

$$f(x) = \begin{cases} x & \text{for } 0 < x < L/2\,, \\ L - x & \text{for } L/2 < x < L \end{cases}$$

for $0 < x < L$. Hence, determine

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}.$$

[8]

(b) Evaluate the Fourier transform of

$$f(x) = \begin{cases} A(a - |x|) & \text{for } -a < x < a, \\ 0 & \text{otherwise.} \end{cases}$$

Here, A and a are some positive constants. Hence, find $(f * f)(x)$ by applying the convolution theorem. *Hint:* You will find the following result useful.

$$\frac{d^4}{dx^4}[x^3 H(x)] = 6\delta(x),$$

where $H(x)$ is the Heaviside step function and $\delta(x)$ is the Dirac delta function.

[7]

Question 2 Sturm-Liouville theory

(a) Consider the set of functions, $\{f(x)\}$, of real variable x, defined in the interval $0 \leq x \leq \pi$, such that $f(0) = f(\pi) = 0$. For unit weight function, determine whether the linear operator

$$\mathcal{L} = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c$$

is Hermitian when acting upon $\{f(x)\}$. Here, a, b and c are some real constants.

[4]

(b) Now consider the second-order differential equation

$$\mathcal{L}y + \lambda y = 0,$$

where $\mathcal{L}$ is as defined in (a), and λ is some real constant. Show how it can be converted into Sturm-Liouville form.

[3]

(c) Hence, solve the Sturm-Liouville eigenvalue equation with $y(0) = y(\pi) = 0$, i.e., find the eigenvalues and corresponding normalized eigenfunctions.

[6]

(d) Show that the eigenfunctions corresponding to different eigenvalues are mutually orthogonal.

[2]

Question 3 Differential multivariable calculus

(a) Consider

$$I(x,y) = \int \tan^{-1}(xy)\,dx.$$

Calculate $\partial I/\partial y$ and hence evaluate $I(x,y)$, or otherwise. *Hint:* you may ignore any constants of integration.

[5]

(b) Find

$$\frac{\partial u}{\partial x}\frac{\partial y}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial y}{\partial v}$$

if

$$u^2 - v^2 = x^3 - 3y, \qquad u + v = 2x + y^2.$$

[5]

(c) At the point $(x,y,z) = (2,2,2)$, find the direction in which the function

$$V(x,y,z) = \exp(x-z)\sin(y-z)$$

has its maximum rate of change and the value of this maximum rate of change.

[2]

(d) Calculate $\nabla \cdot \boldsymbol{F}$ and $\nabla \times \boldsymbol{F}$ when the vector field

$$\boldsymbol{F} = \boldsymbol{F}(x,y,z) = 2xy\boldsymbol{e}_x - z^2\boldsymbol{e}_y + x\boldsymbol{e}_z.$$

[3]

Question 4 Integral multivariable calculus

(a) Let C be the circle of radius 2 about the point with coordinates $(-8,0)$, oriented anticlockwise. Using ***Green's theorem,*** or otherwise, evaluate

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$$

with

$$\boldsymbol{F} = \boldsymbol{F}(x,y) = [\exp(\sin x) - y]\boldsymbol{e}_x + (-4x + \sinh^3 y)\boldsymbol{e}_y.$$

[4]

(b) Evaluate the surface integral

$$\oint_S \boldsymbol{r}\cdot d\boldsymbol{a},$$

where $\boldsymbol{r}$ is the position vector of points on the surface S of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

by parametrising the surface S as $x = a\sin\theta\cos\phi,\ y = b\sin\theta\sin\phi,\ z = c\cos\theta$, with $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. Hence deduce the volume bounded by S, by applying the ***divergence theorem***.

[6]

(c) Evaluate directly and by ***Stokes' theorem*** the line integral

$$\oint_C (y^2dx + z^2dy + x^2dz),$$

if C is the triangle with vertices at $(0,0,0)$, $(0,a,0)$, $(0,0,a)$.

[5]

– End of Paper –

YY