NATIONAL UNIVERSITY OF SINGAPORE

PC2134 MATHEMATICAL METHODS IN PHYSICS I

(Semester I: AY 2018-19)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name.**
2. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
3. Students are required to answer **ALL** questions.
4. Students should write the answers for each question on a new page.
5. This is a **CLOSED** book examination. You are allowed however to bring in **ONE** A4-sized help sheet.
6. Specific permitted devices: non-programmable calculators.

Question 1 **[20=10+10]**

In Hamiltonian mechanics, the central mathematical entity is the Hamiltonian $H(\mathbf{q}(t), \mathbf{p}(t), t)$ which is treated as a function of time t and canonical variables: generalized coordinates q_i and generalized momenta p_i for $i = 1, 2, \ldots, N$.

(a) Suppose that the canonical variables q_i and p_i are functions of two variables u and v, the **Lagrange bracket** $\{u, v\}_{\text{LB}}$ of u and v with respect to $(\mathbf{q}(t), \mathbf{p}(t))$ is defined as

$$\{u, v\}_{\text{LB}} \equiv \sum_{i=1}^{N} \frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u}\frac{\partial q_i}{\partial v} .$$

Evaluate the Lagrange brackets $\{q_m, q_n\}_{\text{LB}}$, $\{p_m, p_n\}_{\text{LB}}$ and $\{q_m, p_n\}_{\text{LB}}$.

(b) The **Poisson bracket** $[F, G]_{\text{PB}}$ of two arbitrary functions $F(\mathbf{q}(t), \mathbf{p}(t))$ and $G(\mathbf{q}(t), \mathbf{p}(t))$ of the canonical variables q_i and p_i for $i = 1, 2, \cdots, N$ is defined as

$$[F, G]_{\text{PB}} \equiv \sum_{i=1}^{N} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} .$$

Consider an arbitrary set of $2N$ *independent* functions $u_k(\mathbf{q}(t), \mathbf{p}(t))$ of canonical variables q_i and p_i for $k = 1, 2, \cdots, 2N$, evaluate

$$\sum_{k=1}^{2N} \{u_k, u_i\}_{\text{LB}} [u_k, u_j]_{\text{PB}} .$$

Question 2 **[20=8+12]**

Consider the following vector function

$$\mathbf{F}(\mathbf{r}) = \left(r^2 \sin^2\theta \, \sin 2\phi + 3\right) \hat{\mathbf{x}} + r\left(r \sin^2\theta \, \cos^2\phi - 4\cos\theta\right) \hat{\mathbf{y}} - 4r \sin\theta \, \sin\phi \, \hat{\mathbf{z}} ,$$

where r, θ and ϕ are spherical coordinates.

(a) Show that $\mathbf{F}(\mathbf{r})$ is a conservative field.

(b) Find a scalar potential $\psi(\mathbf{r})$ such that $\mathbf{F}(\mathbf{r}) = -\nabla\psi(\mathbf{r})$. Hence, or otherwise, evaluate the integral

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} ,$$

where C is the curve from $P_1 = (3, -1, 2)$ to $P_2 = (2, 1, -1)$ expressed in the Cartesian coordinates.

Question 3 **[20=10+10]**

Maxwell's equations take the following form in a homogeneous linear medium:

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = 0\,, \qquad \nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}\,,$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0\,, \qquad \nabla \times \mathbf{B}(\mathbf{r},t) = \mu\epsilon \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} + \mu\sigma \mathbf{E}(\mathbf{r},t)\,,$$

where ϵ, μ and σ are permittivity, permeability and conductivity of the medium.

(a) Show that the electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$ in the medium satisfy the modified wave equations below:

$$\nabla^2 \mathbf{E}(\mathbf{r},t) = \mu\epsilon \frac{\partial^2 \mathbf{E}(\mathbf{r},t)}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}\,, \qquad \nabla^2 \mathbf{B}(\mathbf{r},t) = \mu\epsilon \frac{\partial^2 \mathbf{B}(\mathbf{r},t)}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}\,.$$

(b) These modified wave equations admit a solution in the form of a plane wave propagating in the z direction:

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 \exp\left[\mathrm{i}\left(\tilde{k}z - \omega t\right)\right]\,, \qquad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 \exp\left[\mathrm{i}\left(\tilde{k}z - \omega t\right)\right]\,,$$

where tilded quantities are complex. Here, $\tilde{k}$ is the wave number and ω is the angular frequency. Denoting the wave number as

$$\tilde{k} \equiv k_1 + \mathrm{i}k_2\,,$$

where k_1 and k_2 are real, express k_1 and k_2 in terms of ϵ, μ, σ and ω.

Question 4 **[20=10+10]**

(a) Starting from the linearly independent functions $\psi_n(s) = s^n$, $n = 0, 1, \ldots$, on the range $0 \leq s < \infty$, construct the first three orthonormal functions $\hat{\phi}_0(s)$, $\hat{\phi}_1(s)$ and $\hat{\phi}_2(s)$, with respect to the weight function $w(s) = \mathrm{e}^{-s}$. Hence, expand the function $f(s) = \mathrm{e}^{-s}$ in $\hat{\phi}_n(s)$. Keep the first three terms in the expansion.

(b) Find an eigenfunction expansion for the solution, with boundary conditions $y(0) = y(\pi) = 0$, of the inhomogeneous differential equation

$$\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + \kappa y(x) = f(x)\,,$$

where κ is the constant and

$$f(x) = \begin{cases} x\,, & 0 \leq x \leq \pi/2\,, \\ \pi - x\,, & \pi/2 \leq x \leq \pi\,. \end{cases}$$

Question 5 **[20=8+12]**

The Laplace transform of the function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} \equiv \overline{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t\,,$$

provided that the integral exists.

(a) Given that $H(t-a)$ is the Heaviside step function where $a > 0$, show that

$$\mathcal{L}\{H(t-a)\,f(t-a)\} = \exp(-sa)\,\overline{f}(s)\,.$$

(b) Consider a resistance R and an inductance L connected in series with a voltage $V(t)$. The equation governs the current $I(t)$ in the circuit is

$$L\frac{\mathrm{d}I(t)}{\mathrm{d}t} + I(t)R = V(t)\,.$$

Suppose $I(0) = 0$ and $V(t)$ is a voltage impulse at $t = t_0 > 0$ given by $V(t) = V_0\,\delta(t-t_0)$. Find the current $I(t)$ by the Laplace transform method.

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