NATIONAL UNIVERSITY OF SINGAPORE

PC2230 THERMODYNAMICS AND STATISTICAL MECHANICS

(Semester II: AY 2017-18)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name.**
2. This examination paper contains **EIGHT (8)** questions and comprises **FOUR (4)** printed pages.
3. Students are required to answer **ALL** questions in Part I and **ALL** questions in Part II.
4. Students should write the answers for each question on a new page.
5. This is a **CLOSED** book examination. You are allowed however to bring in **ONE** A4-sized help sheet.
6. Specific permitted devices: non-programmable calculators.

PART I

Question 1 [8=4+4]

The entropy of a two-dimensional gas of particles each with mass m in an area A is given by the expression

$$S(E, A, N) = Nk\left[\ln\left(\frac{A}{N}\right) + \ln\left(\frac{2\pi mE}{Nh}\right) + 2\right],$$

where N is the number of particles and E is the internal energy of the gas.

(a) Find the pressure $P(E, A, N)$ and chemical potential $\mu(E, A, N)$ of the gas.

(b) Find the Helmholtz energy of the gas in terms of its natural variables.

Question 2 [8=5+3]

Consider a composite system of interest (and its constituent subsystems) in contact with a temperature reservoir at temperature T_0, but thermally isolated from the rest of the universe.

(a) Show that the Helmholtz energy of the system is minimized at equilibrium with respect to some extensive thermodynamic variable X of the system.

(b) The composite system is partitioned into two subsystems , $(T^{(1)}, V^{(1)}, N^{(1)})$ and $(T^{(2)}, V^{(2)}, N^{(2)})$, with a fixed wall permeable to particles. Show that when the two subsystems are not in diffusive equilibrium, particles move from the subsystem of higher chemical potential to that of lower one.

Question 3 [8=3+5]

The fundamental equation in the energy representation for a magnetic system is given by

$$E = E(S, M), \qquad \mathrm{d}E = T\,\mathrm{d}S + \mu_0 HV\,\mathrm{d}M,$$

where H is the auxiliary magnetic field, M is the magnetization and μ_0 is the permeability of free space. A particular magnetic material satisfies the following equation of state:

$$M = C\frac{H}{T},$$

where C is a constant.

(a) Calculate the work done when the material is magnetized isothermally from a magnetization M_1 to M_2.

(b) Show that

$$\left(\frac{\partial T}{\partial H}\right)_S = \mu_0 V\frac{CH}{C_H T},$$

where C_H is the constant-field heat capacity.

Question 4 **[8=5+3]**

A system consists of N weakly interacting distinguishable particles. There are two energy levels for each particle: one is non-degenerated ϵ_1 and the other is $\epsilon_2 > \epsilon_1$ with degeneracy g. The occupation of these energy levels are n_1 and n_2 respectively where $N = n_1 + n_2$. The system is in contact with a heat reservoir at temperature T. A simple process occurs in which the occupations change: $n_2 \to n_2 - 1$ and $n_1 \to n_1 + 1$ with the energy released going into the heat reservoir.

(a) Calculate the change in the entropy of the system ΔS_{system} and the change in the entropy of the heat reservoir $\Delta S_{\text{reservoir}}$ respectively.

(b) If the process is reversible, what is the ratio of n_2 to n_1?

Question 5 **[8=3+5]**

A system of N non-interacting identical particles are enclosed in a container of volume V at a temperature T. The Hamiltonian of the system is

$$H_N(\mathbf{x}, \mathbf{p}) = \frac{1}{2m} \sum_{i=1}^{N} p_{i,x}^2 + p_{i,y}^2 + p_{i,z}^2 \,,$$

where m is the mass of the particle and $p_{i,x}$, $p_{i,y}$, $p_{i,z}$ are the Cartesian momenta of the particles.

(a) Derive the Maxwell-Boltzmann velocity distribution $\rho(\mathbf{v})$ from classical canonical ensemble:

$$\rho(\mathbf{v}) = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) .$$

(b) Find the distribution function $\tilde{\rho}(v_\perp)$ for the speed $v_\perp$ for motion *perpendicular* to the z axis and the average speed $\langle v_\perp \rangle$ for motion *perpendicular* to the z-axis.

END OF PART I

PART II

Question 6 **[20=8+12]**

A cylindrical rubber thread is subjected to a tension τ at which an equation of state is empirically obtained as follows:

$$\tau = \alpha\,(L - L_0)\,T\,,$$

where L_0 is the original length of the thread, L is the stretched length and α is a positive constant. The constant-length heat capacity C_L at $L = L_0$ is C which is a positive constant.

(a) Show that the constant-length heat capacity C_L depends neither on the temperature T nor on the length L. Hence, find the internal energy $E(T, L)$ and entropy $S(T, L)$ of the system.

(b) Sketch in the τ-L diagram a Carnot cycle at which the two isotherms belong to the temperatures T_1 and $T_2 > T_1$. In which direction must the cycle be traversed in order to operate as a heat engine? Determine the energy efficiency of this heat engine in terms of T_1 and T_2.

Question 7 **[20=10+10]**

Consider a one dimensional chain composed of $N \gg 1$ located sites. Each site is occupied by a polymer with two energy states: (1) it can be straight with energy 0; or (2) it can bend (on the left or on the right) with energy $\epsilon > 0$ independent of the bending direction.

(a) Using microcanonical ensemble, find the entropy of the system, $S(E, N)$, for a fixed total energy $E = m\epsilon$ where m is an integer such that $m \gg 1$.

(b) Determine the internal energy $E(T, N)$ and the heat capacity $C(T, N)$ of the system. Discuss the behaviour of $E(T, N)$ and $C(T, N)$ in the limit of low and high temperatures.

Question 8 **[20=10+10]**

Consider a system in contact with a temperature reservoir and particle reservoir.

(a) Show that the mean number of particles of the system and its variance are given respectively by

$$\langle N \rangle = -\left(\frac{\partial \Phi}{\partial \mu}\right)_{T,V}, \qquad \langle (\Delta N)^2 \rangle = -kT\left(\frac{\partial^2 \Phi}{\partial \mu^2}\right)_{T,V},$$

where Φ is the grand potential of the system.

(b) Obtain expressions for relative fluctuation $\sqrt{\langle (\Delta N)^2 \rangle}/\langle N \rangle$ for the Bose-Einstein and Fermi-Dirac statistics. Comment on your results.

END OF PART II

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END OF PAPER