Suggested Solution for Statistical Physics and Thermodynamics AY2007-08 Semester 2

NUS Physics Society

1. (a) (Note that the subscript 1, 2 indicating the subsystem 1 and subsystem 2 respectively. Whereas the variables without subscript refer to variables of whole system)

$$S_1 + S_2 = S \qquad V_1 + V_2 = V$$
$$E_1 + E_2 = E \qquad N_1 + N_2 = N \tag{1}$$
$$\text{Hence,} \quad \frac{\partial \mathrm{V}_2}{\partial \mathrm{V}_1} = -1 \quad \text{as} \quad \frac{\partial \mathrm{V}}{\partial \mathrm{V}_1} = 0$$
$$\left(\frac{\partial S_1}{\partial V_1}\right)_{E_1 N_1} = \left(\frac{\partial S_2}{\partial V_2}\right)_{E_2 N_2} \tag{2}$$

The pressure of each subsystem is defined as $P_i = T_i \frac{\partial S_i}{\partial V_i}$ where $i = 1$ or 2.
Hence the equilibrium condition is achieved when $T_1 = T_2$ and $P_1 = P_2$ from equation **??**.

(b)

$$\frac{dS}{dt} = \frac{\partial S_1}{\partial V_1}\frac{dV_1}{dt} + \frac{\partial S_2}{\partial V_2}\frac{dV_2}{dt} = (\frac{P_1}{T_1} - \frac{P_2}{T_2})\frac{dV_1}{dt} > 0 \quad \text{from the second law of thermodynamics.}$$

Hence, if $P_1 > P_2$, $\frac{dV_1}{dt} > 0$ according to the second law.

2. (a)

$$dS = \frac{dQ}{T} \qquad dQ = C \cdot dT$$
$$\Delta S = C\int_{T_1}^{T_0} \frac{1}{T} dT = C\ln(\frac{T_0}{T_1})$$

(b) Change in entropy of water $= mC \cdot ln(\frac{T_0}{T_1}) = 1000 \times 4.2 \times ln(373/273) = 1311 JK^{-1}$
Change in entropy of heat reservoir $= -1311 JK^{-1}$
while change in entropy of the entire system $= 0$

3. (a)

$$S = -\left(\frac{\partial F}{\partial T}\right)_V$$
$$= NK\ln\left[\frac{eV}{N}(\frac{2\pi mkT}{h^2})^{3/2}\right] + \frac{3NKT^{3/2}}{2}$$

(b)

4. (a) Vapour is much less dense than water, $\Delta V = V_{vapour} - V_{\text{liquid}} \approx V_{\text{vapour}}$

$$\begin{aligned} PV &= RT \quad \text{per one mole} \\ \text{Hence, L} &= T\Delta V \frac{dP}{dT} \\ &= \frac{RT^2}{P}\frac{dP}{dT} \end{aligned}$$

(b)

$$\begin{aligned} \frac{dP}{dT} &= \frac{(788 - 733.7)\text{mmHg}}{(373 - 272)\text{K}} \\ &= \frac{54.3\text{mmHg}}{2\text{K}} = 7239\text{PaK}^{-1} \end{aligned}$$

$$\begin{aligned} L &= \frac{RT^2}{P}\frac{dP}{dT} \\ &= \frac{8.31 \times 373^2}{760 \times 133.3} \times 7239 = 82614\text{J/mole} \end{aligned}$$

5. (a) Fermi energy, ϵ_F achived at the condition $T = 0K$. The step function of the equation is ignored. Thus,

$$\begin{aligned} N(\epsilon_F) &= \frac{4\pi V}{h^3}(2m)^{3/2}\int_0^{\epsilon_F} \epsilon^{1/2} d\epsilon \\ &= \frac{4\pi V}{h^3}\frac{2}{3}\epsilon_F^{3/2} \\ \text{Hence, after rearranging,} \quad \epsilon_F &= \frac{h^2}{2m}(\frac{3N}{8\pi V})^{2/3} \end{aligned}$$

(b) $\epsilon_F = k \cdot T_F \qquad \epsilon_F = \frac{1}{2}mV_F^2$ where k is Botlzmann constant.

(c) Given $\frac{N}{V} = 4.7 \times 10^{22} cm^{-3} = 4.7 \times 10^{28} m^{-3}$.

$$\begin{aligned} \epsilon_F &= \frac{h^2}{2m}(\frac{3}{8\pi}\frac{N}{V})^{2/3} \\ &= \frac{(6.626 \times 10^{-34})^2}{2 \times 9.11 \times 10^{-31}}(\frac{3}{8\pi} \times 4.7 \times 10^{28})^{2/3} = 7.608 \times 10^{-19} J \end{aligned}$$

$$T_F = \frac{7.608 \times 10^{-19}}{1.381 \times 10^{-23}} = 5.509 \times 10^4 K$$

$$V_F = \sqrt{\frac{2\epsilon_F}{m}} = \sqrt{\frac{2 \times 7.608 \times 10^{-19}}{9.11 \times 10^{-31}}} = 1.292 \times 10^6 m/s$$

6. Given, $Q_P = kT^2 \frac{d}{dT} lnK_p(T)$

$$\begin{aligned}
Q_V &= Q_P - kT\sum_i v_i = kT^2\frac{d}{dT}\ln K_p(T) - kT\sum_i v_i \\
&= kT^2\frac{d}{dT}\sum_i v_i \ln[kTf_i(T)] - kT\sum_i v_i \\
&= kT^2\frac{d}{dT}\sum_i v_i \ln[f_i(T)] + kT^2\frac{d}{dT}\sum_i v_i \ln[kT] - kT\sum_i v_i \\
&= kT^2\frac{d}{dT}\sum_i v_i \ln[f_i(T)] = kT^2\frac{d}{dT}\ln[\prod_i (f_i(T))^{v_i}] \\
&= kT^2\frac{d}{dT}\ln K_c(T)
\end{aligned}$$

Long Questions

1. (a) The energy of the dipole with magnetic moment parallel to magnetic field, $= -B\cdot\mu$.
The energy of the dipole with magnetic moment anti-parallel to magnetic field, $= B\cdot\mu$.

$$\begin{aligned}
lnZ &= \sum_i exp(-\beta\epsilon_i) = exp(\beta B\mu) + exp(-\beta B\mu) \\
\frac{\partial lnZ}{\partial B} &= \beta\mu(\frac{1}{Z})exp(\beta B\mu) + (-\beta\mu)(\frac{1}{Z})exp(-\beta B\mu) \\
\frac{1}{\beta}(\frac{\partial lnZ}{\partial B})_\beta &= \mu(\frac{1}{Z})exp(\beta B\mu) + (-\mu)(\frac{1}{Z})exp(-\beta B\mu)
\end{aligned}$$

We know that population of dipole with magnetic moment parallel (anti-parallel) to magnetic field $= \frac{1}{Z}exp(\beta B\mu) \quad (\frac{1}{Z}exp(-\beta B\mu))$. Hence,

$$\begin{aligned}
\overline{\mu} &= P_{(+)}\mu + P_{(-)}(-\mu) = \frac{1}{\beta}(\frac{\partial lnZ}{\partial B})_\beta \\
-(\frac{\partial lnZ}{\partial \beta})_B &= (-B\mu)(\frac{1}{Z})exp(\beta B\mu) + (B\mu)(\frac{1}{Z})exp(-\beta B\mu) \\
&= P_{(+)}\epsilon_+ + P_{(-)}\epsilon_- = \overline{E}.
\end{aligned}$$

(b)

$$\begin{aligned}
I &= \frac{N\cdot\overline{\mu}}{V} = \frac{N}{V}\times[\mu(\frac{1}{Z})exp(\beta B\mu) + (-\mu)(\frac{1}{Z})exp(-\beta B\mu)] \\
&= \frac{N}{V}\mu\times\frac{[exp(x) - exp(-x)]}{exp(x) + exp(-x)} \text{ where x = } \beta\text{B}\mu \\
&= \frac{N}{V}\mu\tanh x
\end{aligned}$$

In the limit of low magnetic field and high temperature, $x << 1$ and thus $\tanh x \approx x$. Thus, $I = \frac{N}{V}\mu x = \frac{N\mu^2 B}{VkT}$

Magnetic susceptibility $X = \frac{I}{H} = \frac{N\mu^2\mu_o}{VkT}$ where $H = \frac{B}{\mu_o}$

$X \propto \frac{1}{T}$ Curie's law is verified.

(c) $\Omega(n) = \frac{N!}{n!(N-n)!}$ statistical weight of n dipoles parallel to magnetic field from total N dipoles.

$S(n) = kln[\Omega(n)] = k[Nln(N) - nln(n) - (N-n)ln(N-n)]$

Striling approximation $ln(n!) \approx nln(n) - n$

$$\begin{aligned}
\frac{1}{T} &= (\frac{\partial S}{\partial E}) = \frac{\partial S(n)}{\partial n} \cdot \frac{\partial n}{\partial E} \\
&= \left[k \ln(\frac{N-n}{n})\right]\left[-\frac{1}{2\mu B}\right] asE = (N-2n)\mu B \\
&= \frac{k}{2\mu B}\ln(\frac{n}{N-n})
\end{aligned}$$

Hence, $\beta = \frac{1}{2\mu B}\ln(\frac{n}{N-n})$ and If $n < \frac{1}{2}N$, $T < 0$ (negative temperature)

2. (a) Given $P(v)dv = \frac{4}{\sqrt{\pi}}u^2 exp(-u^2)du = P(u)du$

$$\begin{aligned}
\frac{dP(u)}{du} &= \frac{d}{du}(\frac{4}{\sqrt{\pi}}u^2 exp(-u^2)) \\
&= \frac{-8}{\sqrt{\pi}}u^3 exp(-u^2) + \frac{8}{\sqrt{\pi}}u \cdot exp(-u^2) \\
&= u \cdot exp(-u^2)\frac{8}{\sqrt{\pi}}(1-u^2)
\end{aligned}$$

Hence, when $u = 1$, most probable speed achieved. Which imply that $V_{max} = (\frac{2kT}{m})^{1/2}$.

(b)

$$\begin{aligned}
\overline{v} &= \int_0^\infty vP(v)dv = \int_0^\infty u \cdot (\frac{2kT}{m})^{1/2}\frac{4}{\sqrt{\pi}}u^2 exp(-u^2)du \\
&= (\frac{2kT}{m})^{1/2}\frac{4}{\sqrt{\pi}}\int_0^\infty u^3 exp(-u^2)du \\
&= (\frac{2kT}{m})^{1/2}\frac{4}{\sqrt{\pi}}(\frac{1}{2}) \qquad \text{from identity} \\
&= 2(\frac{2kT}{\pi m})^{1/2}
\end{aligned}$$

(c)

$$\begin{aligned}
V_{rms}^2 &= \int_0^\infty v^2 P(v)dv = \int_0^\infty u^2 \cdot (\frac{2kT}{m})\frac{4}{\sqrt{\pi}}u^2 exp(-u^2)du \\
&= (\frac{2kT}{m})\frac{4}{\sqrt{\pi}}\int_0^\infty u^4 exp(-u^2)du \\
&= (\frac{2kT}{m})\frac{4}{\sqrt{\pi}}(\frac{3}{8}\sqrt{\pi}) \qquad \text{from identity} \\
&= \frac{3kT}{m}
\end{aligned}$$

Hence, $V_{rms} = (\frac{3kT}{m})^{1/2}$

(d)

$$\overline{E} = \frac{1}{2}m \cdot V_{rms}^2 = \frac{3}{2}kT$$

3. (a) Ordinary, the number of particles of BE gas, N is given as

$$V\frac{2\pi(2m)^{3/2}}{h^3}\int_0^\infty \frac{\epsilon^{1/2}d\epsilon}{e^{\beta(\epsilon-\mu)}-1}$$

When, $T\downarrow$, μ must $\uparrow$ for N/V to be kept constant. Meahwhile, $T = T_c$ imply $\mu = 0$. Hence,

$$\frac{N}{V} = (\frac{2\pi mkT_c}{h^2})^{3/2}\left(\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{z^{1/2}dz}{e^z-1}\right) \quad \text{with} \quad z = \frac{\epsilon}{kT_c}$$

Thus, $T_c = \frac{1}{k}(\frac{N}{V})^{2/3}\frac{h^2}{2\pi m}(\frac{1}{2.612})^{2/3}$

(b) Below T_c, the chemical potential is extremely closed to zero. The number of particles with non-zero energy can be computed as

$$N_{\epsilon>0} = V\frac{2\pi(2m)^{3/2}}{h^3}\int_0^\infty \frac{\epsilon^{1/2}d\epsilon}{e^{\beta\epsilon}-1}$$

Introduce $z = \beta\epsilon$,$N_{\epsilon>0} = V(\frac{2\pi mkT}{h^2})^{3/2}\left(\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{z^{1/2}dz}{ez-1}\right) = N(\frac{T}{T_c})^{3/2}$

Meanwhile, for number of particles with zero-energy ground state is $N(1-(\frac{T}{T_c})^{3/2})$.

(c) Energy of BE gas at $T < T_c$, and introduce $z = \beta\epsilon$

$$\begin{aligned} E &= V(\frac{2\pi mkT}{h^2})^{3/2}\left(\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{z^{3/2}dz}{e^z-1}\right) \\ N &= V(\frac{2\pi mkT_c}{h^2})^{3/2}\left(\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{z^{1/2}dz}{e^z-1}\right) \end{aligned}$$

Hence, $E = 0.77Nk\frac{T^{5/2}}{T_c^{3/2}}$ computed with the identity given by question. Heat capacity, $C_v = \frac{dE}{dT} = \frac{5}{2}\times 0.77NR(\frac{T}{T_c})^{3/2} = 1.93NR(\frac{T}{T_c})^{3/2}$

(d) Bose-Einstein condensation is different from ordinary vapour-liquid condensation in the way that no *spatial* separation into phases in BE gas.