AY 2006/07, Sem 2, PC3130, Exam

Question. . .1/6. . .

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[1] Since $\vec{V} = \frac{d}{dt}\vec{R} = \frac{1}{i\hbar}[\vec{R}, H]$ and the expectation value is taken in an eigenstate of H, we have

$$\langle \vec{V} \rangle = 0, \text{ indeed.}$$

[2] For the harmonic oscillator we have (page 73)

$$X(t) = X(0)\cos\phi + \frac{1}{M\omega}P(0)\sin\phi,$$

$$P(t) = P(0)\cos\phi - M\omega X(0)\sin\phi, \text{ with } \phi = \omega t;$$

and for $\psi(x) = \sqrt{\kappa}\, e^{-\kappa|x|}$, which is real and even in x, we have

$$\langle X \rangle = 0, \quad \langle P \rangle = 0, \quad \langle (XP + PX) \rangle = 0,$$

and also $\langle X^2 \rangle = \int_{-\infty}^{\infty} dx\, x^2 \kappa e^{-2\kappa|x|}$

$$= 2\kappa \int_0^{\infty} dx\, x^2 e^{-2\kappa x} = 2\kappa \frac{2!}{(2\kappa)^3} = \frac{1}{2\kappa^2}$$

as well as

$$\langle P^2 \rangle = \int_{-\infty}^{\infty} dx\, \kappa \left(\hbar\kappa e^{-\kappa|x|}\right)^2 = (\hbar\kappa)^2.$$

(a) Obviously $\langle X(t) \rangle = 0$ and $\langle P(t) \rangle = 0$.
Further

$$\langle X(t)^2 \rangle = \langle X(0)^2 \rangle (\cos\phi)^2 + \left(\frac{1}{M\omega}\right)^2 \langle P(0)^2 \rangle (\sin\phi)^2 + \frac{1}{M\omega}\langle (X(0)P(0) + P(0)X(0)) \rangle \cos\phi \sin\phi$$

$$= \frac{1}{2\kappa^2}(\cos\phi)^2 + \left(\frac{\hbar\kappa}{M\omega}\right)^2 (\sin\phi)^2$$

and

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$$\langle P(t)^2\rangle = \langle P(0)^2\rangle(\cos\phi)^2 + (M\omega)^2\langle X(0)^2\rangle(\sin\phi)^2$$
$$- M\omega\langle (P(0)X(0)+X(0)P(0))\rangle\cos\phi\sin\phi$$
$$= (\hbar k)^2(\cos\phi)^2 + (M\omega)^2\frac{1}{2k^2}(\sin\phi)^2.$$

Therefore,

$$\delta X(t) = \sqrt{\langle X(t)^2\rangle - \langle X(t)\rangle^2}$$
$$= \sqrt{\frac{1}{2k^2} + \left[\left(\frac{\hbar k}{M\omega}\right)^2 - \frac{1}{2k^2}\right](\sin\phi)^2}$$
$$= \frac{1}{\sqrt{2}\,k}\sqrt{1 + \left[2\left(\frac{\hbar k^2}{M\omega}\right)^2 - 1\right](\sin\phi)^2}$$

and

$$\delta P(t) = \sqrt{\langle P(t)^2\rangle - \langle P(t)\rangle^2}$$
$$= \sqrt{(\hbar k)^2 + \left[\frac{1}{2}\left(\frac{M\omega}{k}\right)^2 - (\hbar k)^2\right](\sin\phi)^2}$$
$$= \hbar k\sqrt{1 + \left[\frac{1}{2}\left(\frac{M\omega}{\hbar k^2}\right)^2 - 1\right](\sin\phi)^2}.$$

(b) We have $\delta X(t) = \delta X(0)\sqrt{1 + (\epsilon^2 - 1)(\sin\phi)^2}$

and $\delta P(t) = \delta P(0)\sqrt{1 + \left(\frac{1}{\epsilon^2} - 1\right)(\sin\phi)^2}$

with $\epsilon^2 = 2\left(\frac{\hbar k^2}{M\omega}\right)^2$.

so that $[\delta X(t)\,\delta P(t)]^2 = [\delta X(0)\,\delta P(0)]^2$

$$\times\left[1 + \left(\epsilon - \frac{1}{\epsilon}\right)^2(\sin\phi\cos\phi)^2\right].$$

It follows that $\delta X(t)\,\delta P(t) \geq \delta X(0)\,\delta P(0)$

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$= \frac{1}{\sqrt{2}K} \hbar K = \frac{\hbar}{2}\sqrt{2} > \frac{\hbar}{2}$ so that

Heisenberg's relation is obeyed, indeed.

(c) There is no t dependence if there is no ϕ dependence which happens only if $\epsilon^2 = 1$ or $K = \sqrt{\frac{M\omega}{\sqrt{2}\hbar}}$.

$\boxed{3}$ (a)
$$\frac{d}{dt} L_1 = \frac{1}{i\hbar}[L_1, H]$$
$$= \frac{1}{i\hbar}\frac{1}{2I_2}\underbrace{[L_1, L_2^2]}_{= i\hbar(L_2 L_3 + L_3 L_2)} + \frac{1}{i\hbar}\frac{1}{2I_3}\underbrace{[L_1, L_3^2]}_{= -i\hbar(L_2 L_3 + L_3 L_2)}$$
$$= \left(\frac{1}{2I_2} - \frac{1}{2I_3}\right)(L_2 L_3 + L_3 L_2).$$

(b) The state ket is $|l=1, m=1\rangle \equiv |1,1\rangle$, for which
$$L_1 |1,1\rangle = |1,0\rangle \frac{\hbar}{\sqrt{2}},$$
$$L_1^2 |1,1\rangle = (|1,1\rangle + |1,-1\rangle)\frac{\hbar^2}{2},$$
$$L_2 |1,1\rangle = |1,0\rangle \frac{i\hbar}{\sqrt{2}},$$
$$L_2^2 |1,1\rangle = (|1,1\rangle - |1,-1\rangle)\frac{\hbar^2}{2},$$
$$L_3^2 |1,1\rangle = |1,1\rangle \hbar^2,$$

so that

$$H|1,1\rangle = |1,1\rangle\left(\frac{\hbar^2}{4I_1} + \frac{\hbar^2}{4I_2} + \frac{\hbar^2}{2I_3}\right) + |1,-1\rangle\left(\frac{\hbar^2}{4I_1} - \frac{\hbar^2}{4I_2}\right)$$

and we read off that

$$\langle H\rangle = \frac{\hbar^2}{4I_1} + \frac{\hbar^2}{4I_2} + \frac{\hbar^2}{2I_3},$$

$$\delta H = \left|\frac{\hbar^2}{4I_1} - \frac{\hbar^2}{4I_2}\right|.$$

(c) For $I_2 = I_3$, we have $H = \frac{1}{2I_1}L_1^2 + \frac{1}{2I_2}(L_2^2 + L_3^2)$

or $H = \frac{1}{2I_1}L_1^2 + \frac{1}{2I_2}(\vec{L}^2 - L_1^2)$

so that the common eigenstates of $\vec{L}^2$ and L_1 are the eigenstates of H. The eigenvalues are

$$H = \frac{\hbar^2}{2I_2}\,l(l+1) + \left(\frac{1}{2I_1} - \frac{1}{2I_2}\right)(\hbar m)^2$$

with $l = 0, 1, 2, \ldots$ and $m = 0, \pm 1, \ldots, \pm l$.

$\boxed{4}$ (a) This probability is the squared amplitude for the $Z=1$ and $Z=2$ ground states:

$$\text{prob. amplitude} = \int (d\vec{r})\, \langle Z=2, \underset{n}{1}\underset{l}{0}\underset{m}{0} | Z=1, \underset{n}{1}\underset{l}{0}\underset{m}{0}\rangle$$

$$= \int (d\vec{r})\, \underbrace{\left(R_{10}(r)Y_{00}\right)^*}_{Z=2}\, \underbrace{\left(R_{10}Y_{00}\right)}_{Z=1}$$

$$= \frac{1}{4\pi}\int (d\vec{r})\;\; 2\left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0}\;\; 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$$

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$$= 4\left(\frac{2}{a_0^2}\right)^{3/2}\int_0^\infty dr\, r^2\, e^{-3r/a_0} = 4\left(\frac{2}{a_0^2}\right)^{3/2}\frac{2!}{(3/a_0)^3}$$

$$= \frac{2^{9/2}}{3^3}\ ;\ \text{probability} = \frac{2^9}{3^6} = \underline{\underline{\left(\frac{8}{9}\right)^3}}.$$

(b) Now we go from $|Z=1, 100\rangle$ to $\langle Z=2, 200|$:

$$\text{Prob. amplitude} = \frac{1}{4\pi}\int (d\vec{r})\, \underbrace{R_{20}(r)^*}_{Z=2}\, \underbrace{R_{10}(r)}_{Z=1}$$

$$= \int_0^\infty dr\, r^2 \left(\frac{1}{a_0}\right)^{3/2}\left(\frac{2r}{a_0}-2\right)e^{-r/a_0}\, 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$$

$$= \frac{4}{a_0^3}\int_0^\infty dr\, r^2\left(\frac{r}{a_0}-1\right)e^{-2r/a_0}$$

$$= \frac{4}{a_0^3}\left(\frac{3!}{16}-\frac{2!}{8}\right)a_0^3 = \frac{1}{2}\ ;\ \text{probability} = \underline{\underline{\frac{1}{4}}}.$$

(c) He we have the transition from the $l=0$ ground state to the $l=1$ excited state, so that the wave fuctions involve orthogonal spherical harmonics and the probability amplitude vanishes; probability $\underline{\underline{=0}}$.

$\boxed{5}$ We have $\langle m^{(0)}|H_1|n^{(0)}\rangle = \hbar\Omega\left(\langle m^{(0)}|(n+1)^{(0)}\rangle + \langle (m+1)^{(0)}|n^{(0)}\rangle\right)$

$$= \hbar\Omega\left(\delta_{m,n+1}+\delta_{m+1,n}\right).$$

(a) $E_n \cong E_n^{(0)} + E_n^{(1)} + E_n^{(2)}$

$$= \hbar\omega n + 0 - \sum_{m(\neq n)}\frac{\left[\hbar\Omega\left(\delta_{m,n+1}+\delta_{m+1,n}\right)\right]^2}{\hbar\omega\,(m-n)}$$

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so that for $\underline{n=0}$:

$$E_0 \cong -\frac{(\hbar\Omega)^2}{\hbar\omega}$$ (only $m=1$ contributes)

and for $n = 1, 2, 3, \ldots$:

$$E_n \cong n\hbar\omega \,.$$ ($m=n+1$ and $m=n-1$ contribute equal amounts with opposite sign)

(b) $$E_0 = E_0^{(0)} + E_0^{(1)} - \sum_{m>0} \frac{|\langle m^{(0)}|H_1|0^{(0)}\rangle|^2}{m\hbar\omega - E_0}$$

$$= 0 + 0 - \frac{(\hbar\Omega)^2}{\hbar\omega - E_0}$$

or $E_0(E_0 - \hbar\omega) = (\hbar\Omega)^2$, which gives

$$E_0 = +\frac{1}{2}\hbar\omega - \sqrt{\left(\tfrac{1}{2}\hbar\omega\right)^2 + (\hbar\Omega)^2} \,.$$

We do not have "+" here because we need to get $E_0 \to E_0^{(0)} = 0$ for $\Omega \to 0$.