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Question 1(a)

$l^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle$
$l_3|l,m\rangle = m\hbar|l,m\rangle$
$l^2 = l_1^2 + l_2^2 + l_3^2$

Since $[l^2, l_\pm] = 0$,
$l^2 l_+|l,m\rangle = l_+ l^2|l,m\rangle = l_+ l(l+1)\hbar^2|l,m\rangle = l(l+1)\hbar^2(l_+|l,m\rangle)$
$l^2 l_-|l,m\rangle = l(l+1)\hbar^2(l_-|l,m\rangle)$

$l_3 l_+|l,m\rangle = (l_+ l_3 + l_+\hbar)|l,m\rangle = l_+(m+1)\hbar|l,m\rangle$
$l_3 l_-|l,m\rangle = l_-(m-1)\hbar|l,m\rangle$

$\therefore\ l_+|l,m\rangle \propto |l,m+1\rangle, \qquad l_-|l,m\rangle \propto |l,m-1\rangle$

We let $l_+|l,m\rangle = c_+|l,m+1\rangle$, where c_+ is an eigenvalue of l_+.
$\langle l,m|l_- l_+|l,m\rangle = |c_+|^2\langle l,m+1|l,m+1\rangle = |c_+|^2$
$l_- l_+ = l^2 - \hbar l_3 - l_3^2$

$$\begin{aligned}\langle l,m|l_- l_+|l,m\rangle &= \langle l,m|l^2 - \hbar l_3 - l_3^2|l,m\rangle \\ &= l(l+1)\hbar^2 - m\hbar^2 - m^2\hbar^2 \\ &= [l(l+1) - m(m+1)]\hbar^2 \\ &= |c_+|^2\end{aligned}$$

$\therefore\ c_+ = \sqrt{l(l+1) - m(m+1)}\hbar$

Similarly,
$\langle l,m|l_+ l_-|l,m\rangle = |c_-|^2, \qquad l_+ l_- = l^2 + \hbar l_3 - l_3^2$
$c_- = \sqrt{l(l+1) - m(m-1)}\hbar$

$\therefore\ l_+|l,m\rangle = \sqrt{l(l+1) - m(m+1)}\hbar|l,m+1\rangle$
$l_-|l,m\rangle = \sqrt{l(l+1) - m(m-1)}\hbar|l,m-1\rangle$

$$l_1 = \frac{1}{2}(l_+ + l_-)$$
$$\langle l,m|l_1|l,m\rangle = \frac{1}{2}\langle l,m|l_+ + l_-|l,m\rangle = 0$$
$$l_1^2 = \frac{1}{4}(l_+ + l_-)^2 = \frac{1}{4}(l_+^2 + l_+ l_- + l_- l_+ + l_-^2)$$

$$\begin{aligned}\langle l,m|l_1^2|l,m\rangle &= \frac{1}{4}\langle l,m|l_+^2 + l_+ l_- + l_- l_+ + l_-^2|l,m\rangle \\ &= \frac{\hbar^2}{4}[l(l+1) - m(m+1) + l(l+1) - m(m-1)] \\ &= \frac{1}{2}[l(l+1) - m^2]\hbar^2\end{aligned}$$

Question 1(b)(i)

$$\Psi(\vec{x},t)=\frac{1}{\sqrt{2}}\left[-\frac{1}{\sqrt{\pi a_0}}\frac{re^{-\frac{r}{2a_0}}}{8a_0^2}\sin\theta\, e^{i\phi}+\frac{1}{\sqrt{\pi a_0}}\frac{re^{-\frac{r}{2a_0}}}{8a_0^2}\sin\theta\, e^{-i\phi}\right]e^{i\frac{e^2}{8a_0\hbar}t}$$

$$=-\frac{ire^{-\frac{r}{2a_0}}\sin\theta\sin\phi}{2\sqrt{\pi}a_0^{\frac{5}{2}}}e^{-\frac{r}{2a_0}+i\frac{e^2}{8a_0\hbar}t}$$

Question 1(b)(ii)

$$V=-\frac{\mathrm{e}^2}{r}$$

$$\langle V\rangle=\langle\Psi|V|\Psi\rangle$$

$$=-\int\frac{\sin^2\theta\sin^2\phi}{4\pi a_0^5}r^2e^{-\frac{r}{a_0}}\frac{\mathrm{e}^2}{r}dV$$

$$=-\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{r^3\mathrm{e}^2\sin^3\theta\sin^2\phi}{4\pi a_0^5}e^{-\frac{r}{a_0}}d\phi\,d\theta\,dr$$

$$=-\frac{\mathrm{e}^2}{4\pi a_0^5}\int_0^\infty a_0^4\left(\frac{r}{a_0}\right)^3e^{-\frac{r}{a_0}}d\left(\frac{r}{a_0}\right)\int_0^\pi\sin^3\theta\,d\theta\int_0^{2\pi}\sin^2\phi\,d\phi$$

$$=-\frac{\mathrm{e}^2}{4\pi a_0^5}(6a_0^4)\left[\frac{2(2)^2}{6}\right](\pi)$$

$$=-\frac{2\mathrm{e}^2}{a_0}$$

Question 2(a)(i)

$$\left(\frac{p^2}{2m}+\frac{1}{2}m\omega^2r^2\right)\Psi(x)=E_n\Psi(x)$$

$$\left[\frac{p_x^2+p_y^2+p_z^2}{2m}+\frac{1}{2}m\omega^2(x^2+y^2+z^2)\right]X(x)Y(y)Z(z)=E_nX(x)Y(y)Z(z)$$

$$\frac{1}{X}\left(-\frac{\hbar^2}{2m}\frac{\partial^2X}{\partial x^2}+\frac{1}{2}m\omega^2x^2\right)+\frac{1}{Y}\left(-\frac{\hbar^2}{2m}\frac{\partial^2Y}{\partial y^2}+\frac{1}{2}m\omega^2y^2\right)+\frac{1}{Z}\left(-\frac{\hbar^2}{2m}\frac{\partial^2Z}{\partial z^2}+\frac{1}{2}m\omega^2z^2\right)$$
$$=E_x+E_y+E_z$$

$$E_n=E_x+E_y+E_z=\left(\frac{3}{2}+n\right)\hbar\omega,\qquad n=n_x+n_y+n_z$$

Question 2(a)(ii)

Degeneracy,

$l=n,n-2,n-4,\dots,0$ or 1

$$C_{k-1}^{n+k-1}=C_n^{n+2}=\frac{(n+2)!}{n!\,2!}=\frac{1}{2}(n+2)(n+1)$$

Question 2(a)(iii)
$\langle r,\theta,\phi|n,l,m\rangle = f(r)Y_l^m(\theta,\phi)$

Under space inversion, $\vec{x} \to -\vec{x}$
$(r,\theta,\phi) \to (r,\pi-\theta,\phi+\pi)$
$Y_l^m(\theta,\phi) \to (-1)^l Y_l^m(\theta,\phi)$
$\therefore$ parity is $(-1)^l$

Question 2(b)(i)
Rotating a scalar quantity,

$$S \to U^\dagger S U = \left(1 + i\frac{\varepsilon\vec{J}}{\hbar}\right) S \left(1 - i\frac{\varepsilon\vec{J}}{\hbar}\right) = S + \left[i\frac{\varepsilon\vec{J}}{\hbar}, S\right]$$

Since S is unchanged in rotation, $\left[i\frac{\varepsilon\vec{J}}{\hbar}, S\right] = [\vec{J}, S] = 0$

For the vector K_i,

$$K_i \to U^\dagger K_i U = \left(1 + i\frac{\varepsilon J_z}{\hbar}\right) K_i \left(1 - i\frac{\varepsilon J_z}{\hbar}\right) = K_i + \frac{i\varepsilon}{\hbar}[J_z, K_i]$$

The normal rotational matrix shows $R_z = \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} \Rightarrow R_z \begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} = \begin{pmatrix} K_x - \varepsilon K_y \\ K_y + \varepsilon K_x \\ K_z \end{pmatrix}$$

Comparing with the above, we see that
$[J_z, K_x] = -i\hbar K_y, \qquad [J_z, K_y] = i\hbar K_x, \qquad [J_z, K_z] = 0$
$\therefore [J_i, K_j] = i\epsilon_{ijk}\hbar K_k$

Question 2(b)(ii)

Question 3

$$J_1^2|j_1,m_1\rangle = 2\hbar^2|j_1,m_1\rangle = j_1(j_1+1)\hbar^2|j_1,m_1\rangle$$
$$j_1^2 + j_1 - 2 = 0 \quad \Rightarrow \quad j_1 = -2,1$$

$$J_2^2|j_2,m_2\rangle = \frac{3}{4}\hbar^2|j_2,m_2\rangle = j_2(j_2+1)\hbar^2|j_2,m_2\rangle$$
$$j_2^2 + j_2 - \frac{3}{4} = 0 \quad \Rightarrow \quad j_2 = -\frac{3}{2},\frac{1}{2}$$

So we have $j_1 = 1, j_2 = \frac{1}{2}$. Using $|j,m\rangle \Rightarrow |j_1,j_2,m_1,m_2\rangle$, we get

$$\left|\frac{3}{2},\frac{3}{2}\right\rangle = \left|1,\frac{1}{2},1,\frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \left|1,\frac{1}{2},-1,-\frac{1}{2}\right\rangle$$

$$J_- = J_{1-} + J_{2-}$$
$$\begin{aligned} J_-|j,m\rangle &= \sqrt{j(j+1)-m(m-1)}|j,m-1\rangle \\ &= J_{1-}|j_1,j_2,m_1,m_2\rangle + J_{2-}|j_1,j_2,m_1,m_2\rangle \\ &= \sqrt{j_1(j_1+1)-m_1(m_1-1)}|j_1,j_2,m_1-1,m_2\rangle + \sqrt{j_2(j_2+1)-m_2(m_2-1)}|j_1,j_2,m_1,m_2-1\rangle \end{aligned}$$

$$J_-\left|\frac{3}{2},\frac{3}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{2}\left|1,\frac{1}{2},0,\frac{1}{2}\right\rangle + \left|1,\frac{1}{2},1,-\frac{1}{2}\right\rangle$$
$$\therefore \left|\frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|1,\frac{1}{2},0,\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|1,\frac{1}{2},1,-\frac{1}{2}\right\rangle$$

$$\begin{aligned} J_-\left|\frac{3}{2},\frac{1}{2}\right\rangle &= 2\left|\frac{3}{2},-\frac{1}{2}\right\rangle \\ &= \sqrt{\frac{2}{3}}(\sqrt{2})\left|1,\frac{1}{2},-1,\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}(\sqrt{2})\left|1,\frac{1}{2},0,-\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}(1)\left|1,\frac{1}{2},0,-\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}(0)\left|1,\frac{1}{2},1,-\frac{3}{2}\right\rangle \end{aligned}$$
$$\therefore \left|\frac{3}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|1,\frac{1}{2},-1,\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|1,\frac{1}{2},0,-\frac{1}{2}\right\rangle$$

Now we find $|j_1-1,j_2-1\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle$. $\quad \left(j=\frac{1}{2},m=\frac{1}{2}\right)$

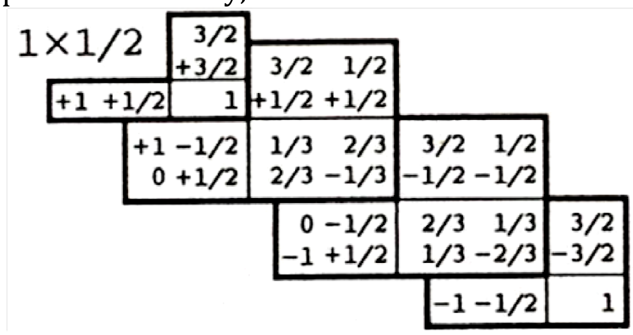
$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \alpha\left|0,\frac{1}{2},0,\frac{1}{2}\right\rangle + \beta\left|0,\frac{1}{2},1,-\frac{1}{2}\right\rangle$$
$$|\alpha|^2 + |\beta|^2 = 1$$
$$\sqrt{\frac{2}{3}}\alpha + \sqrt{\frac{1}{3}}\beta = 0$$
$$\Rightarrow \alpha = -\sqrt{\frac{1}{3}}, \beta = \sqrt{\frac{2}{3}}$$
$$\therefore \left|\frac{1}{2},\frac{1}{2}\right\rangle = -\sqrt{\frac{1}{3}}\left|1,\frac{1}{2},0,\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|1,\frac{1}{2},1,-\frac{1}{2}\right\rangle$$

$$J_-\left|\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$= -\sqrt{\frac{1}{3}}(\sqrt{2})\left|1,\frac{1}{2},-1,\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}(\sqrt{2})\left|1,\frac{1}{2},0,-\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}(1)\left|1,\frac{1}{2},0,-\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}(0)\left|1,\frac{1}{2},1,-\frac{3}{2}\right\rangle$$
$$\therefore \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|0,\frac{1}{2},0,-\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|0,\frac{1}{2},-1,\frac{1}{2}\right\rangle$$

To summarize, the complete set of simultaneous normalized eigenfunctions are:

$$\left|\frac{3}{2},\frac{3}{2}\right\rangle = \left|1,\frac{1}{2},1,\frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|1,\frac{1}{2},0,\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|1,\frac{1}{2},1,-\frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|1,\frac{1}{2},-1,\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|1,\frac{1}{2},0,-\frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \left|1,\frac{1}{2},-1,-\frac{1}{2}\right\rangle$$
$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = -\sqrt{\frac{1}{3}}\left|1,\frac{1}{2},0,\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|1,\frac{1}{2},1,-\frac{1}{2}\right\rangle$$
$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|0,\frac{1}{2},0,-\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|0,\frac{1}{2},-1,\frac{1}{2}\right\rangle$$

p.s.: Alternatively, one can check the Clebsch-Gordan coefficient table:



Question 4(a)(i)

In quantum physics, since identical particles lose their individuality, there may be more than one ket or bra vectors that can represent the same physical state of the system of identical particles. According to postulate 1, each physical state is completely described by a ket or bra vector in Hilbert space. This is not a problem for individual particles, but for more than 2 particles. Since particles are indistinguishable, we find that we have 2 mathematical states describing the same physical state, since we could obtain a 2nd mathematical state by exchanging the quantum numbers of the particles. This condition is known as exchange degeneracy.

It can be removed by introducing Postulate VII, insisting that states must be symmetrized or anti-symmetrized with respect to any permutation of N identical particles.

Question 4(a)(ii)
Consider the Hamiltonian of a pair of particle and its anti-particle, e.g. electron and positron This Hamiltonian commutes with the permutation operator, but the states need not be symmetrized or anti-symmetrized.

Question 4(b)(i)
For distinguishable particles, $\psi(x_1, x_2) = \psi_n(x_1)\psi_l(x_2)$.

Also, $\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_n + \langle x^2 \rangle_l - 2\langle x \rangle_n \langle x \rangle_l$, since

$$\langle x \rangle_n = \int_0^a \psi_n^*(x) x \psi_n(x) dx = \frac{2}{a}\int_0^a x \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{a}{2}$$

$$\langle x^2 \rangle_n = \int_0^a \psi_n^*(x) x^2 \psi_n(x)\, dx = \frac{2}{a}\int_0^a x^2 \sin^2\left(\frac{n\pi x}{a}\right) dx = a^2\left[\frac{1}{3} - \frac{1}{2(n\pi)^2}\right]$$

$$\therefore \langle (x_1 - x_2)^2 \rangle = a^2\left[\frac{1}{3} - \frac{1}{2(n\pi)^2}\right] + a^2\left[\frac{1}{3} = \frac{1}{2(l\pi)^2}\right] - 2\frac{a}{2}\left(\frac{a}{2}\right) = a^2\left(\frac{1}{6} - \frac{1}{2n^2\pi^2} - \frac{1}{2l^2\pi^2}\right)$$

Question 4(b)(ii)
If they are identical bosons, they wave function is:

$$\psi_+(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_n(x_1)\psi_l(x_2) + \psi_l(x_1)\psi_n(x_2)]$$

Then the expectation value of $(x_1 - x_2)^2$ is:

$$\langle (x_1 - x_2)^2 \rangle_+ = \int_0^a \psi_+^*(x_1 - x_2)^2 \psi_+\, dx_1 dx_2$$

Evaluate $\psi_+^*\psi_+$:

$$\begin{aligned}\psi_+^*\psi_+ &= \frac{1}{2}[\psi_n^*(x_1)\psi_l^*(x_2) + \psi_l^*(x_1)\psi_n^*(x_2)][\psi_n(x_1)\psi_l(x_2) + \psi_l(x_1)\psi_n(x_2)] \\ &= \frac{1}{2}[\psi_n^*(x_1)\psi_l^*(x_2)\psi_n(x_1)\psi_l(x_2) + \psi_n^*(x_1)\psi_l^*(x_2)\psi_l(x_1)\psi_n(x_2) \\ &\qquad + \psi_l^*(x_1)\psi_n^*(x_2)\psi_n(x_1)\psi_l(x_2) + \psi_l^*(x_1)\psi_n^*(x_2)\psi_l(x_1)\psi_n(x_2)]\end{aligned}$$

Working out the expectation value of $(x_1 - x_2)^2$:

$$\begin{aligned}\langle (x_1 - x_2)^2 \rangle_+ &= \int_0^a (x_1 - x_2)^2 \psi_+^*\psi_+\, dx_1 dx_2 \\ &= \int_0^a (x_1 - x_2)^2[\psi_n^*(x_1)\psi_l^*(x_2)\psi_n(x_1)\psi_l(x_2) + \psi_n^*(x_1)\psi_l^*(x_2)\psi_l(x_1)\psi_n(x_2)]\, dx_1 dx_2 \\ &= \langle (x_1 - x_2)^2 \rangle_{dist} + \int_0^a (x_1^2 + x_2^2 - 2x_1x_2)[\psi_n^*(x_1)\psi_l^*(x_2)\psi_l(x_1)\psi_n(x_2)]\, dx_1 dx_2 \\ &= \langle (x_1 - x_2)^2 \rangle_{dist} - \int_0^a 2x_1x_2[\psi_n^*(x_1)\psi_l^*(x_2)\psi_l(x_1)\psi_n(x_2)]\, dx_1 dx_2 \\ &= \langle (x_1 - x_2)^2 \rangle_{dist} - 2\int_0^a x_1[\psi_n^*(x_1)\psi_l(x_1)]\, dx_1 \int_0^a x_2[\psi_l^*(x_2)\psi_n(x_2)]\, dx_2 \\ &= \langle (x_1 - x_2)^2 \rangle_{dist} - 2\int_0^a \frac{4}{a^2}\sin\left(\frac{n\pi x_1}{a}\right) x_1 \sin\left(\frac{l\pi x_1}{a}\right) dx_1 \int_0^a \frac{4}{a^2}\sin\left(\frac{n\pi x_2}{a}\right) x_2 \sin\left(\frac{l\pi x_2}{a}\right) dx_2 \\ &= \langle (x_1 - x_2)^2 \rangle_{dist} - 2\left[\int_0^a \frac{4}{a^2}\sin\left(\frac{n\pi x}{a}\right) x \sin\left(\frac{l\pi x}{a}\right) dx\right]^2 \\ &= \langle (x_1 - x_2)^2 \rangle_{dist} - 2|\langle x \rangle_{nl}|^2\end{aligned}$$

$$= a^2\left(\frac{1}{6}-\frac{1}{2n^2\pi^2}-\frac{1}{2l^2\pi^2}\right)-2\left\{\frac{a}{\pi^2}[(-1)^{n+l}-1]\left[\frac{1}{(n-l)^2}-\frac{1}{(n+l)^2}\right]\right\}^2$$
$$= a^2\left\{\frac{1}{6}-\frac{1}{2\pi^2}\left(\frac{1}{n^2}-\frac{1}{l^2}\right)-\frac{32n^2l^2}{\pi^4(n^2-l^2)^4}[(-1)^{n+l}-1]^2\right\}$$

Question 4(b)(iii)

If they are identical fermions, then the wave function is

$$\psi_-(x_1,x_2)=\frac{1}{\sqrt{2}}[\psi_n(x_1)\psi_l(x_2)-\psi_l(x_1)\psi_n(x_2)]$$

Then the expectation value of $(x_1-x_2)^2$ is:

$$\langle(x_1-x_2)^2\rangle_-=\int_0^a\psi_-^*(x_1-x_2)^2\psi_-\,dx_1dx_2$$

Evaluate $\psi_-^*\psi_-$:

$$\psi_-^*\psi_-=\frac{1}{2}[\psi_n^*(x_1)\psi_l^*(x_2)\psi_n(x_1)\psi_l(x_2)-\psi_n^*(x_1)\psi_l^*(x_2)\psi_l(x_1)\psi_n(x_2)$$
$$-\psi_l^*(x_1)\psi_n^*(x_2)\psi_n(x_1)\psi_l(x_2)+\psi_l^*(x_1)\psi_n^*(x_2)\psi_l(x_1)\psi_n(x_2)]$$

The expectation value of $(x_1-x_2)^2$:

$$\langle(x_1-x_2)^2\rangle_-=\langle(x_1-x_2)^2\rangle_{dist}+2|\langle x\rangle_{nl}|^2$$
$$= a^2\left\{\frac{1}{6}-\frac{1}{2\pi^2}\left(\frac{1}{n^2}-\frac{1}{l^2}\right)+\frac{32n^2l^2}{\pi^4(n^2-l^2)^4}[(-1)^{n+l}-1]^2\right\}$$

Solutions provided by:

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* John Soo (Q1, Q2, Q4a)