NATIONAL UNIVERSITY OF SINGAPORE

PC3130 Quantum Mechanics II

(Semester II: AY 2008-09)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **four** questions and comprises **4** printed pages.
2. Answer **any three** questions.
3. Answers to the questions are to be written in the answer books.
4. This is a CLOSED BOOK examination.

1. (a) The operators $\underset{\sim}{\ell}^2$ and ℓ_3 describe the square and the third component of the orbital angular operator $\underset{\sim}{\ell}$ respectively. Let $|\ell\, m\rangle$ be their common eigenstate.

Show that

$$\ell_+ |\ell\, m\rangle = \sqrt{\ell(\ell+1) - m(m+1)}\ \hbar\ |\ell\ m+1\rangle$$
$$\ell_- |\ell\, m\rangle = \sqrt{\ell(\ell+1) - m(m-1)}\ \hbar\ |\ell\ m-1\rangle .$$

Here the operators ℓ_+ and ℓ_- are defined by $\quad \ell_\pm = \ell_1 \pm i\ell_2$.

Hence show that the expectation value of ℓ_1 in the state $|\ell\, m\rangle$ vanishes but that of $\ell_1{}^2$ is given by

$$\langle \ell\, m|\ell_1{}^2|\ell\, m\rangle = \frac{1}{2}\,[\ell(\ell+1) - m^2\,]\,\hbar^2$$

(b) A hydrogen atom starts out in the following linear combination of the stationary states $n=2, \ell=1, m=1$ and $n=2,\ \ell=1, m=-1$, with energy $E_2 = -e^2/(8a_0)$ and a_0 is the Bohr radius ($\sim 0.5\times10^{-10}$ m):

$$\Psi(\underset{\sim}{x},0) = \frac{1}{\sqrt{2}}\left(\psi_{211} + \psi_{21-1}\right),$$

where $\psi_{21\pm1} = \mp\dfrac{1}{\sqrt{\pi\, a_0}}\dfrac{1}{8a_0^2}\, r\, e^{-r/(2a_0)}\, \sin\theta\, e^{\pm i\phi}$ in the usual notations.

(i) Construct the state at time $t, \Psi(\underset{\sim}{x},t)$. Simplify it as much as you can.

(ii) Calculate the expectation value of the potential energy, $\langle V\rangle$.

2. (a) Consider the three-dimensional harmonic oscillator, for which the potential is $V(r) = \frac{1}{2}m\omega^2 r^2$, where m is the mass of the particle and ω a positive constant.

(i) Show that separation of variables in Cartesian coordinates turns

this into three one-dimensional oscillators, and hence or otherwise, determine the allowed energy.
You may assume without proof that the energy of a one-dimensional harmonic oscillator is $\left(n+\frac{1}{2}\right)\hbar\,\omega$.

(ii) Obtain a formula for the degeneracy of the n^{th} excited stationary state. What are the allowed values of the orbital angular momentum quantum number ℓ?

(iii) What is the parity of the n^{th} excited stationary state?

(b)(i) Define a scalar operator S and a vector operator K_i in quantum mechanics.
Deduce the commutation relations of the angular momentum operator J_i with the scalar operator S and with the vector operator K_i, respectively. Does S commute with K_i?
You can assume without proof that for an infinitesimal angle ε, the rotation $\Re_{\underset{\sim}{n}}(\varepsilon)$ in three-dimensional physical space is given by $\Re_{\underset{\sim}{n}}(\varepsilon) = (1 + \varepsilon \underset{\sim}{n} \wedge)$. The rotation operator in the Hilbert space can be written as $R = \exp\left(-i\theta \underset{\sim}{n}.\underset{\sim}{J}/\hbar\right)$ in the usual notations.

(ii) Write down and simplify the matrix element of the scalar operator S in the angular momentum basis $|\tau\, j m\rangle$.

3. Obtain from first principles a complete set of simultaneous normalized eigenvectors of $\underset{\sim}{J}_1^2$, $\underset{\sim}{J}_2^2$, $\underset{\sim}{J}^2$ and J_3, where $\underset{\sim}{J} = \underset{\sim}{J}_1 + \underset{\sim}{J}_2$, given simultaneous normalized eigenvectors $|j_1 m_1>$ and $|j_2 m_2>$ of $\underset{\sim}{J}_1^2$, J_{13} and $\underset{\sim}{J}_2^2$, J_{23} respectively such that

$$\underset{\sim}{J}_1^2 |j_1 m_1> = 2\hbar^2 |j_1 m_1>,$$

$$\underset{\sim}{J}_2^2 |j_2 m_2> = \frac{3}{4}\hbar^2 |j_2 m_2>.$$

4. (a)(i) What is exchange degeneracy and how is it removed?

(ii) The Hamiltonian of a physical system commutes with the permutation operator U_p, where p is a transposition. Explain whether this necessarily implies that the physical states of the system must be symmetrized or antisymmetrized .

(b) Two noninteracting particles, each of mass m, are in an infinite square well, $V(x)=0$ for $0<x<a$ and $V(x)=\infty$ otherwise. Suppose one of the particles is in the state $\psi_n(x)=\sqrt{2/a}\,\sin(n\pi x/a)$, and the other particle in state ψ_l $(l \neq n)$.
Write down the state of the two-particle system and compute the average value of their separation, namely $\langle (x_1 - x_2)^2 \rangle$. In your calculation, you should distinguish three cases: (i) the particles are distinguishable, (ii) the particles are identical bosons, and (iii) the particles are identical fermions.

The following results can be used without proof:

$$\langle x \rangle_n = \int_0^a \psi_n^*(x)\, x\, \psi_n(x)\, dx = \frac{2}{a}\int_0^a x \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{a}{2}$$

$$\langle x^2 \rangle_n = \int_0^a \psi_n^*(x)\, x^2\, \psi_n(x)\, dx = a^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2}\right)$$

$$\langle x \rangle_{nl} = \int_0^a \psi_n^*(x)\, x\, \psi_l(x)\, dx = \frac{a}{\pi^2}\left[(-)^{n+l} - 1\right]\left(\frac{1}{(n-l)^2} - \frac{1}{(n+l)^2}\right)$$

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