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Question 1(a)

Time independent Schrödinger Equation, for a particle moving in a central field:

$$H = \frac{p^2}{2m} + V(r)$$
$$\left[\frac{p^2}{2m} + V(r)\right]\Psi = E\Psi$$

Use separation of variable, $\Psi = f(r)Y_l^m(\theta,\phi)$

$$\frac{p^2}{2m} + V(r) \Rightarrow \frac{p_r^2}{2m} + \frac{l^2}{2mr^2} + V(r) = \left(-\frac{i\hbar}{r}\frac{\partial}{\partial r}r\right)^2 + \frac{l^2}{2mr^2} + V(r)$$

Let $f(r) = \frac{R(r)}{r}$. Then,

$$\left[\left(-\frac{i\hbar}{r}\frac{\partial}{\partial r}r\right)^2 + \frac{l^2}{2mr^2} + V(r)\right]R(r)Y_l^m(\theta,\phi) = ER(r)Y_l^m(\theta,\phi)$$
$$\left[-\frac{\hbar^2}{2mr}\frac{\partial^2}{\partial r^2}r + \frac{l(l+1)\hbar^2}{2mr^2} + V(r)\right]R(r) = ER(r)$$

So we have the radial equation,

$$\left[-\frac{\hbar^2}{2mr}\frac{\partial^2}{\partial r^2}r + \frac{l(l+1)\hbar^2}{2mr^2} + V(r) - E\right]R(r) = 0$$
$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2}(E-V)\right]R(r) = 0$$

Question 1(b)

Given $\left[\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} - \frac{1}{4} + \frac{n}{\rho}\right] y(\rho) = 0$

For $\rho \to 0$, the equation is $\left[\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2}\right] y(\rho) = 0$, the acceptable solution is $y(\rho) = \rho^{l+1}$.

For $\rho \to \infty$, the equation is $\left[\frac{d^2}{d\rho^2} - \frac{1}{4}\right] y(\rho) = 0$, the acceptable solution is $y(\rho) = e^{-\frac{\rho}{2}}$.

Thus the solution for $y(\rho)$ can be written as $y(\rho) = \rho^{l+1} V(\rho) e^{-\frac{\rho}{2}}$.

As $\frac{dy}{d\rho} = (l+1)\rho^l V e^{-\frac{\rho}{2}} + \rho^{l+1} V' e^{-\frac{\rho}{2}} - \frac{1}{2}\rho^{l+1} V e^{-\frac{\rho}{2}}$ etc, we get the differential equation for V as

$\rho V'' + (2l + 2 - \rho)V' + (n - l - 1)V = 0$

Let $V(\rho) = \sum_{j=0}^{\infty} c_j \rho^j$, $\quad V'(\rho) = \sum_{j=1}^{\infty} c_j j \rho^{j-1}$, $\quad V''(\rho) = \sum_{j=2}^{\infty} c_j j(j-1)\rho^{j-2}$,

substituting into the differential equation,

$$\rho \sum_{j=2}^{\infty} c_j j(j-1)\rho^{j-2} + (2l+2)\sum_{j=1}^{\infty} c_j j \rho^{j-1} - \sum_{j=1}^{\infty} c_j j \rho^j + (n-l-1)\sum_{j=0}^{\infty} c_j \rho^j = 0$$

Comparing the coefficients of ρ^j, we get

$(j+1)j c_{j+1} + (2l+2)(j+1)c_{j+1} - j c_j + (n-l-1)c_j = 0$

$\therefore c_{j+1} = \frac{l+1+j-n}{(j+1)(2l+2+j)} c_j \quad$ [shown]

For $l = n$, $\quad c_{j+1} = \frac{1+j}{(j+1)(2n+2+j)} c_j$

$c_1 = \frac{1}{2n+2} c_0, \quad c_0 = 0 \;\Rightarrow\; V(\rho) = 0$

$\therefore y(\rho) = \rho^{n+1} V(\rho) e^{-\frac{\rho}{2}} = 0$

Question 1(c)

This question can be solved by using Taylor expansion. First find the value of r, r_0 when $V(r)$ is minimum, by setting $V'(r) = 0$. Next, rewrite the value of $V(r)$ as an expansion of $(r - r_0)$,

$$V(r) \approx V(r_0) + V'(r_0)(r - r_0) + \frac{1}{2}V''(r_0)(r - r_0)^2 + \cdots$$

After that, $V(r)$ would have the form of r^2. Substitute $V(r)$ into the time-independent Schrödinger's equation, and it can be solved just the same method a simple harmonic oscillator is solved.

Question 2(a)(i)

Given $|\psi'\rangle = U|\psi\rangle$, $|\phi'\rangle = U|\phi\rangle$. Transition probability after symmetry transformation U is given by $|\langle\phi'|\psi'\rangle|^2$. Since $|\phi'\rangle = U|\phi\rangle$, $\langle\phi'| = \langle\phi|U^\dagger$, we have $|\langle\phi'|\psi'\rangle|^2 = \left|\langle\phi|U^\dagger U|\psi\rangle\right|^2$. After the symmetrical transition, the transition probability has to be equal to $|\langle\phi|\psi\rangle|^2$. Since $|\langle\phi'|\psi'\rangle|^2 = |\langle\phi|\psi\rangle|^2$, we get $|\langle\phi|\psi\rangle|^2 = \left|\langle\phi|U^\dagger U|\psi\rangle\right|^2$, for any $|\phi\rangle$ and $|\psi\rangle$.

$\therefore U^\dagger U = 1 \quad \Rightarrow \quad U^\dagger = U^{-1}$. U is unitary.

Question 2(a)(ii)

Consider $\langle\phi'|\omega'\rangle$, with $|\phi'\rangle = U|\phi\rangle$, $|\omega'\rangle = U|\omega\rangle$. So $\langle\phi'|\omega'\rangle = \langle\phi'|U|\omega\rangle$. We let $|\omega\rangle = \alpha|\psi\rangle$, where α is a constant. We then have $\langle\phi'|\omega'\rangle = \langle\phi'|U\alpha|\psi\rangle$. As transition probability is preserved, $|\langle\phi'|\psi'\rangle|^2 = |\langle\phi|\psi\rangle|^2$, we have $\langle\phi'|\omega'\rangle\langle\phi'|\omega'\rangle^* = \langle\phi|\omega\rangle\langle\phi|\omega\rangle^*$.

We now have 2 possibilities:

(1) If $\langle\phi'|\omega'\rangle = \langle\phi|\omega\rangle$, then RHS $= \langle\phi|\omega\rangle = \langle\phi|\alpha|\psi\rangle = \alpha\langle\phi'|\psi'\rangle = \alpha\langle\phi'|U|\psi\rangle = \langle\phi'|\alpha U|\psi\rangle$. Then LHS $= \langle\phi'|\omega'\rangle = \langle\phi'|U\alpha|\psi\rangle = \langle\phi'|\alpha U|\psi\rangle =$ RHS. Since $|\phi'\rangle$ is any arbitrary state vector, $U\alpha|\psi\rangle = \alpha U\,|\psi\rangle$, U is linear.

(2) If $\langle\phi'|\omega'\rangle = \langle\phi|\omega\rangle^*$, then RHS $= \langle\phi|\omega\rangle^* = \langle\phi|\alpha|\psi\rangle^* = \alpha^*\langle\phi|\psi\rangle^* = \alpha^*\langle\phi'|\psi'\rangle = \alpha^*\langle\phi'|U|\psi\rangle = \langle\phi'|\alpha^* U|\psi\rangle$. Then LHS $= \langle\phi'|\omega'\rangle = \langle\phi'|U\alpha|\psi\rangle = \langle\phi'|\alpha^* U|\psi\rangle =$ RHS. So we have $U\alpha|\psi\rangle = \alpha^* U\,|\psi\rangle$, U is anti-linear.

$\therefore$ U is linear or anti-linear.

Question 2(a)(iii)

Suppose $|\psi'\rangle = U|\psi\rangle$, then if $|\psi\rangle$ and $|\psi'\rangle$ are dynamically possible, then

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle, \qquad i\hbar\frac{\partial}{\partial t}|\psi'\rangle = H|\psi'\rangle$$

$$i\hbar\frac{\partial}{\partial t}U|\psi\rangle = HU|\psi\rangle$$

$$i\hbar\frac{\partial U}{\partial t}|\psi\rangle + i\hbar U\frac{\partial}{\partial t}|\psi\rangle = HU|\psi\rangle$$

$$i\hbar\frac{\partial U}{\partial t}|\psi\rangle + UH|\psi\rangle = HU|\psi\rangle$$

$$i\hbar\frac{\partial U}{\partial t}|\psi\rangle + [U,H]|\psi\rangle = 0$$

$\therefore$ If U doesn't depend on time explicitly, $\dfrac{\partial U}{\partial t} = 0$, then $[U,H]|\psi\rangle = 0$, $[U,H] = 0$.

Question 2(b)(i)

$$\frac{d}{dt}\langle\vec{l}\rangle = \frac{i}{\hbar}\langle[H,\vec{l}]\rangle + \langle\frac{\partial\vec{l}}{\partial t}\rangle = \frac{i}{\hbar}\langle[H,\vec{l}]\rangle$$

Since $\vec{l}$ doesn't depend on time explicitly.

The Hamiltonian of a single particle is $H = \frac{p^2}{2m} + V(\vec{x})$. We know that $[p^2,\vec{l}] = 0$.

$$\begin{aligned}[V(\vec{x}),\vec{l}] &= \epsilon_{ijk}[V(\vec{x}),x_j p_k] \\ &= \epsilon_{ijk}x_j[V(\vec{x}),p_k] + \epsilon_{ijk}[V(\vec{x}),x_j]p_k \\ &= \epsilon_{ijk}x_j[V(\vec{x}),p_k] \\ &= \epsilon_{ijk}x_j[V(\vec{x})p_k - p_k V(\vec{x})]\end{aligned}$$

$$[V(\vec{x}),\vec{l}]|\psi\rangle = \epsilon_{ijk}x_j V(\vec{x})p_k|\psi\rangle - \epsilon_{ijk}x_j p_k[V(\vec{x})|\psi\rangle]$$
$$= -\epsilon_{ijk}x_j p_k V(\vec{x})|\psi\rangle$$

$$[V(\vec{x}),\vec{l}] = -\epsilon_{ijk}x_j p_k V(\vec{x}) = -(\vec{x}\times\vec{p})V(\vec{x}) = -\vec{l}V(\vec{x})$$

$$\therefore [H,\vec{l}] = [V(\vec{x}),\vec{l}] = -\vec{l}V(\vec{x}) = \frac{\hbar}{i}\vec{N}$$
$$\therefore \frac{d}{dt}\langle\vec{l}\rangle = \frac{i}{\hbar}\langle[H,\vec{l}]\rangle = \langle\vec{N}\rangle \quad \text{[shown]}$$

Question 2(b)(ii)

For a spherical potential, we have $[\vec{l},H] = 0$.

$$\therefore \frac{d}{dt}\langle\vec{l}\rangle = \frac{i}{\hbar}\langle[H,\vec{l}]\rangle + \langle\frac{\partial\vec{l}}{\partial t}\rangle = 0 \quad \text{[shown]}$$

Question 3(a)

Infinitesimal rotation operator in 3-D physical space,

$\mathcal{R} = 1 + \varepsilon\hat{n}\times$

$\vec{x} \to \vec{x}', \qquad \mathcal{R}\vec{x} = \vec{x} + \varepsilon\hat{n}\times\vec{x}$

Consider the behavior of the wavefunction under rotation in 3D. For infinitesimal rotation,

$\psi_i'(\vec{x}) = \pi_{ij}\psi_i(\vec{x}') = \pi_{ij}\psi_i(\mathcal{R}^{-1}\vec{x}') = \pi_{ij}\psi_i(\vec{x}' - \varepsilon\hat{n}\times\vec{x}')$

Using Taylor's expansion,

$$\begin{aligned}\psi(\vec{x}' - \varepsilon\hat{n}\times\vec{x}') &= \psi(\vec{x}') - (\varepsilon\hat{n}\times\vec{x}')\cdot\vec{\nabla}\psi(\vec{x}') + \cdots \\ &= \psi(\vec{x}') - \frac{i}{\hbar}(\varepsilon\hat{n}\times\vec{x}'\cdot\vec{p})\psi(\vec{x}') + \cdots \\ &= \psi(\vec{x}') - \frac{i}{\hbar}(\varepsilon\hat{n}\cdot\vec{x}'\times\vec{p})\psi(\vec{x}') + \cdots \\ &= \psi(\vec{x}') - \frac{i}{\hbar}(\varepsilon\hat{n}\cdot\vec{l})\psi(\vec{x}') + \cdots \\ &= \left(1 - \frac{i}{\hbar}\varepsilon\hat{n}\cdot\vec{l}\right)\psi(\vec{x}') + \cdots\end{aligned}$$

Writing $\pi = 1 - \frac{i}{\hbar}\varepsilon\hat{n}\cdot\vec{s}$, we get

$$\begin{aligned}\psi'(\vec{x}) &= \pi\psi(\vec{x} - \varepsilon\hat{n}\times\vec{x}) \\ &= \left(1 - \frac{i}{\hbar}\varepsilon\hat{n}\cdot\vec{s}\right)\left[\left(1 - \frac{i}{\hbar}\varepsilon\hat{n}\cdot\vec{l}\right)\psi(\vec{x}) + \cdots\right] \\ &= \left[1 - \frac{i}{\hbar}\varepsilon\hat{n}\cdot\left(\vec{s} + \vec{l}\right)\right]\psi(\vec{x}) + \cdots \\ &\approx \left[1 - \frac{i}{\hbar}\varepsilon\hat{n}\cdot\vec{J}\right]\psi(\vec{x}) \\ &= R_{\hat{n}}(\varepsilon)\psi(\vec{x})\end{aligned}$$

$$\therefore R_{\hat{n}}(\varepsilon) = 1 - \frac{i}{\hbar}\varepsilon\hat{n}\cdot\vec{J} \quad \text{[shown]}$$

For finite rotation, $R_{\hat{n}}(\theta + \delta\theta) = R_{\hat{n}}(\theta)R_{\hat{n}}(\delta\theta)$. But $R_{\hat{n}}(\delta\theta) = 1 - \frac{i}{\hbar}\delta\theta\hat{n}\cdot\vec{J}$. So we have

$$R_{\hat{n}}(\theta + \delta\theta) = R_{\hat{n}}(\theta)\left(1 - \frac{i}{\hbar}\delta\theta\hat{n}\cdot\vec{J}\right)$$

$$\frac{R_{\hat{n}}(\theta + \delta\theta) - R_{\hat{n}}(\theta)}{\delta\theta} = -\frac{i}{\hbar}\hat{n}\cdot\vec{J}R_{\hat{n}}(\theta)$$

$$\frac{d}{d\theta}R_{\hat{n}}(\theta) = -\frac{i}{\hbar}\hat{n}\cdot\vec{J}R_{\hat{n}}(\theta)$$

$$R_{\hat{n}}(\theta) = e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{J}} \quad \text{[shown]}$$

Question 3(b)

$$
\begin{aligned}
e^{-\frac{i}{\hbar}\theta(\hat{n}\cdot\vec{s})} &= e^{-\frac{i}{\hbar}\theta\left(\hat{n}\cdot\frac{\hbar}{2}\vec{\sigma}\right)} \\
&= e^{-i\left(\frac{\theta}{2}\right)(\hat{n}\cdot\vec{\sigma})} \\
&= 1 - i\left(\frac{\theta}{2}\right)(\hat{n}\cdot\vec{\sigma}) - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2(\hat{n}\cdot\vec{\sigma})^2 + i\frac{1}{3!}\left(\frac{\theta}{2}\right)^3(\hat{n}\cdot\vec{\sigma})^3 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4(\hat{n}\cdot\vec{\sigma})^4 - i\frac{1}{5!}\left(\frac{\theta}{2}\right)^5(\hat{n}\cdot\vec{\sigma})^5 \\
&\qquad + \cdots \\
&= 1 - i\left(\frac{\theta}{2}\right)(\hat{n}\cdot\vec{\sigma}) - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + i\frac{1}{3!}\left(\frac{\theta}{2}\right)^3(\hat{n}\cdot\vec{\sigma}) + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 - i\frac{1}{5!}\left(\frac{\theta}{2}\right)^5(\hat{n}\cdot\vec{\sigma}) + \cdots \\
&= \left[1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 + \cdots\right] - i(\hat{n}\cdot\vec{\sigma})\left[\left(\frac{\theta}{2}\right) - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \frac{1}{5!}\left(\frac{\theta}{2}\right)^5 + \cdots\right] \\
&= \cos\frac{\theta}{2} - i\hat{n}\cdot\vec{\sigma}\sin\frac{\theta}{2} \quad \text{[shown]}
\end{aligned}
$$

Question 3(c)

$$
\begin{aligned}
\hat{n}\cdot\vec{\sigma} &= n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3 \\
&= \sin\theta\cos\phi\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \sin\theta\sin\phi\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \cos\theta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \\
&= \begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta\\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta\\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}\\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix} &= \begin{pmatrix}e^{-i\frac{\phi}{2}}\cos\theta\cos\frac{\theta}{2} + e^{i\frac{\phi}{2}}e^{-i\phi}\sin\theta\sin\frac{\theta}{2}\\ e^{i\phi}e^{-i\frac{\phi}{2}}\sin\theta\cos\frac{\theta}{2} - e^{i\frac{\phi}{2}}\cos\theta\sin\frac{\theta}{2}\end{pmatrix} \\
&= \begin{pmatrix}e^{-i\frac{\phi}{2}}\left(\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}\right)\\ e^{i\frac{\phi}{2}}\left(\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}\right)\end{pmatrix} \\
&= -\begin{pmatrix}e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}\\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix}
\end{aligned}
$$

$\therefore \begin{pmatrix}e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}\\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix}$ is an eigenstate of $\hat{n}\cdot\vec{\sigma}$.

Question 3(d)

Question 4(a)

$H|j_1,j_2,m_1,m_2\rangle = E|j_1,j_2,m_1,m_2\rangle$

$\left[\frac{E_1}{\hbar^2}(\vec{J}_1+\vec{J}_2)\cdot\vec{J}_2+\frac{E_2}{\hbar^2}(\vec{J}_{13}+\vec{J}_{23})^2\right]|j_1,j_2,m_1,m_2\rangle = E|j_1,j_2,m_1,m_2\rangle$

Since $[\vec{J}_{13},\vec{J}_{23}]=0$, we have $\vec{J}_{13}\cdot\vec{J}_{23}+\vec{J}_{23}\cdot\vec{J}_{13}=2\vec{J}_{13}\cdot\vec{J}_{23}$.

Then since $[\vec{J}_1,\vec{J}_2]=0$, $J^2=(J_1+J_2)^2=J_1^2+2\vec{J}_1\cdot\vec{J}_2+J_2^2$, we get $\vec{J}_1\cdot\vec{J}_2=\frac{1}{2}(J^2-J_1^2-J_2^2)$.

From the above information, we arrive at

$\left[\frac{E_1}{2\hbar^2}(J^2-J_1^2+J_2^2)+\frac{E_2}{\hbar^2}(J_{13}^2+2\vec{J}_{13}\cdot\vec{J}_{23}+J_{23}^2)\right]|j_1,j_2,m_1,m_2\rangle = E|j_1,j_2,m_1,m_2\rangle$

$\left[\frac{E_1}{2}[j(j+1)-j_1(j_1+1)+j_2(j_2+1)]+E_2(m_1^2+2m_1m_2+m_2^2)\right]|j_1,j_2,m_1,m_2\rangle = E|j_1,j_2,m_1,m_2\rangle$

When $j_1=j_2=1$, the energy levels are

$E=\frac{1}{2}E_1j(j+1)+E_2(m_1+m_2)^2$

To find the energy eigenstates, we refer to the Clebsch-Gordan table. For $|j,m\rangle \Rightarrow |j_1,j_2,m_1,m_2\rangle$, the eigenstates and their respective energies can be summarized by the table below:

Eigenstate	Energy
$\|2,2\rangle = \|1,1,1,1\rangle$	$3E_1+4E_2$
$\|2,1\rangle = \frac{1}{\sqrt{2}}\|1,1,1,0\rangle + \frac{1}{\sqrt{2}}\|1,1,0,1\rangle$	$3E_1+E_2$
$\|1,1\rangle = \frac{1}{\sqrt{2}}\|1,1,1,0\rangle - \frac{1}{\sqrt{2}}\|1,1,0,1\rangle$	E_1+E_2
$\|2,0\rangle = \frac{1}{\sqrt{6}}\|1,1,1,-1\rangle + \sqrt{\frac{2}{3}}\|1,1,0,0\rangle + \frac{1}{\sqrt{6}}\|1,1,-1,1\rangle$	$3E_1$
$\|1,0\rangle = \frac{1}{\sqrt{2}}\|1,1,1,-1\rangle - \frac{1}{\sqrt{2}}\|1,1,-1,1\rangle$	E_1
$\|0,0\rangle = \frac{1}{\sqrt{3}}\|1,1,1,-1\rangle - \frac{1}{\sqrt{3}}\|1,1,0,0\rangle + \frac{1}{\sqrt{3}}\|1,1,-1,1\rangle$	0
$\|2,-1\rangle = \frac{1}{\sqrt{2}}\|1,1,0,-1\rangle + \frac{1}{\sqrt{2}}\|1,1,-1,0\rangle$	$3E_1+E_2$
$\|1,-1\rangle = \frac{1}{\sqrt{2}}\|1,1,0,-1\rangle - \frac{1}{\sqrt{2}}\|1,1,-1,0\rangle$	E_1+E_2
$\|2,-2\rangle = \|1,1,-1,-1\rangle$	$3E_1+4E_2$

The degeneracy for the states whose total angular momentum quantum number $j=2$ is 3, because there are 3 energy levels: $3E_1+4E_2$, $3E_1+E_2$, and $3E_1$.

Question 4(b)

Energy for 2 particle system,

$$E_{n,k} = \left(n + k + \frac{1}{2}\right)\hbar\omega, \qquad n, k = 0,1,2, \ldots$$

<u>Case 1: Identical Bosons</u>

Eigenfunction,

$$\psi_+(X) = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\left[\frac{H_k(X_1)H_n(X_2)}{\sqrt{k!\,n!\,2^{k+n}}}e^{-\frac{1}{2}(X_1^2+X_2^2)} + \frac{H_k(X_2)H_n(X_1)}{\sqrt{k!\,n!\,2^{k+n}}}e^{-\frac{1}{2}(X_1^2+X_2^2)}\right]$$

$$= \frac{e^{-\frac{1}{2}(X_1^2+X_2^2)}}{\sqrt{k!\,n!\,2^{k+n+1}}}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}[H_k(X_1)H_n(X_2) + H_k(X_2)H_n(X_1)]$$

Ground state, $k = 0, n = 0$

$$\psi_+ = e^{-\frac{1}{2}(X_1^2+X_2^2)}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\sqrt{2}H_0(X_1)H_0(X_2), \qquad E_{00} = \frac{1}{2}\hbar\omega$$

First excited state, $k = 1, n = 0$

$$\psi_+ = \frac{e^{-\frac{1}{2}(X_1^2+X_2^2)}}{2}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}[H_0(X_1)H_1(X_2) + H_0(X_2)H_1(X_1)], \qquad E_{01} = \frac{3}{2}\hbar\omega$$

Second excited state, $k = 1, n = 1$

$$\psi_+ = \frac{e^{-\frac{1}{2}(X_1^2+X_2^2)}}{\sqrt{2}}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}H_1(X_1)H_1(X_2), \qquad E_{11} = \frac{5}{2}\hbar\omega$$

<u>Case 2: Identical Fermions</u>

Eigenfunction,

$$\psi_-(X) = \frac{e^{-\frac{1}{2}(X_1^2+X_2^2)}}{\sqrt{k!\,n!\,2^{k+n+1}}}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}[H_k(X_1)H_n(X_2) - H_k(X_2)H_n(X_1)]$$

Ground state, $k = 1, n = 0$

$$\psi_- = \frac{e^{-\frac{1}{2}(X_1^2+X_2^2)}}{2}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}[H_1(X_1)H_0(X_2) - H_1(X_2)H_0(X_1)], \qquad E_{01} = \frac{3}{2}\hbar\omega$$

First excited state, $k = 2, n = 0$

$$\psi_- = \frac{e^{-\frac{1}{2}(X_1^2+X_2^2)}}{\sqrt{2}}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}[H_2(X_1)H_0(X_2) - H_2(X_2)H_0(X_1)], \qquad E_{02} = \frac{5}{2}\hbar\omega$$

Second excited state, $k = 1, n = 2$ or $k = 3, n = 0$

$$\psi_- = \frac{e^{-\frac{1}{2}(X_1^2+X_2^2)}}{4}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}[H_1(X_1)H_2(X_2) - H_1(X_2)H_2(X_1)], \qquad E_{12} = \frac{7}{2}\hbar\omega$$

$$\psi_- = \frac{e^{-\frac{1}{2}(X_1^2+X_2^2)}}{4\sqrt{3}}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}[H_3(X_1)H_0(X_2) - H_3(X_2)H_0(X_1)], \qquad E_{03} = \frac{7}{2}\hbar\omega$$

Ground state wavefunction of 3 identical fermions in the same potential well:

$k = 0, n = 1, m = 2$

$$\psi_-(X) = \frac{e^{-\frac{1}{2}(X_1^2+X_2^2+X_3^2)}}{4\sqrt{6}}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}[H_0(X_1)H_1(X_2)H_2(X_3) + H_1(X_1)H_2(X_2)H_0(X_3) + H_2(X_1)H_0(X_2)H_1(X_3) - H_0(X_1)H_2(X_2)H_1(X_3) - H_2(X_1)H_1(X_2)H_0(X_3) - H_1(X_1)H_0(X_2)H_2(X_3)]$$

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