NATIONAL UNIVERSITY OF SINGAPORE

PC3130 Quantum Mechanics II

(Semester II: AY 2009-10)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **4** questions and comprises **7** printed pages, including a Table of the Clebsch-Gordan coefficients.
2. Answer **any 3** questions.
3. All questions carry equal marks.
4. Answers to the questions are to be written in the answer books.
5. This is a CLOSED BOOK examination.

1. (a) For a particle moving in a central potential field $V(r)$, write down the time- independent Schrodinger equation.

Making use of the spherical symmetry, **briefly** outline the steps that reduce this equation to a one-dimensional radial equation.

Note: The following relations can be used without proof:

(i) $\underset{\sim}{p}^2 = p_r^2 + \frac{\underset{\sim}{\ell}^2}{r^2}$, $p_r = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}r$, (ii) $\underset{\sim}{\ell}^2\, Y_\ell^m(\theta,\phi) = \ell(\ell+1)\hbar^2\, Y_\ell^m(\theta,\phi)$

(b) The energy eigenfunction of a three-dimensional simple harmonic oscillator with energy E and angular momentum (ℓ, m) can be written as

$$\psi_{n\ell m}(\underset{\sim}{x}) = \frac{y(r)}{r}\, Y_\ell^m(\theta,\phi).$$

Here n is the principal quantum number related to the energy E, and $Y_\ell^m(\theta,\phi)$ is the spherical harmonics. The radial function $y(r)$ satisfies the following differential equation

$$\left[\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} + \frac{2E}{\hbar\omega} - \rho^2\right] y(\rho) = 0.$$

where $\rho^2 = \frac{m\omega}{\hbar} r^2$.

Show that $y(\rho)$ can be expressed as

$$y(\rho) = \rho^{\ell+1} V(\rho) e^{-\rho^2/2}.$$

Obtain the differential equation for the function $V(\rho)$ and find its solution for the case $\ell = n$.

(c) The simplest molecular crystals are formed from noble gases such as neon, argon, krypton and xenon. The interaction between the ions in such a molecular crystal is approximated by the Lennard-Jones potential

$$V(r) = 4V_0\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$

where the distance $r=\sqrt{x_1^2+x_2^2+x_3^2}$, V_0 and σ are the parameters for the noble gases, e g for krypton, $V_0=0.014eV$ and $\sigma=36.5nm$.

Find approximately the ground state energy of a single ion in such a crystal.

Hint: The ion near the minimum value of the potential energy $V(r)$ can be treated as a harmonic oscillator, the energy of a 3-dimensional harmonic oscillator is $\hbar\omega(n+3/2)$ in the usual notations.

2. **(a)** Define a symmetry transformation U in quantum mechanics.

Show that the symmetry transformation U has the following properties:

(i) U is unitary, i.e. $U^+ = U^{-1}$

(ii) U is linear or anti-linear: $U(\lambda|\psi\rangle) = \lambda\, U|\psi\rangle$ or $U(\lambda|\psi\rangle) = \lambda^*\, U|\psi\rangle$.

(iii) If U does not depend on time explicitly, then $[U,H]=0$

where H is the Hamiltonian of the physical system under consideration.

(b)(i) Prove that, for a particle in a potential $V(\underset{\sim}{x})$, the rate of change of the expectation value of the orbital angular momentum $\underset{\sim}{\ell}$ is equal to the expectation value of the torque

$$\frac{d}{dt}\langle\underset{\sim}{\ell}\rangle=\langle\underset{\sim}{N}\rangle,$$

where $\underset{\sim}{N} = -\frac{i}{\hbar}\underset{\sim}{\ell}\underset{\sim}{V}(x) = -\frac{i}{\hbar}\underset{\sim}{x} \wedge \left(\underset{\sim}{p}\underset{\sim}{V}(x)\right) = \underset{\sim}{x} \wedge \left(-\nabla V(\underset{\sim}{x})\right)$.

Note: The following relation can be used without proof:

$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle [H,Q]\rangle + \left\langle \frac{\partial Q}{\partial t}\right\rangle$ where Q is any operator and H is the Hamiltonian.

(ii)Show that $\frac{d}{dt}\left\langle \underset{\sim}{\ell}\right\rangle = 0$ for any spherically symmetric potential.

3. **(a)** Write down the infinitesimal rotation operator in three-dimensional physical space and show that the corresponding infinitesimal rotation operator in Hilbert space can be written as

$$R_{\underset{\sim}{n}}(\varepsilon) = 1 - i\varepsilon \underset{\sim}{n}\cdot \underset{\sim}{J}/\hbar$$

where $\underset{\sim}{J} = \underset{\sim}{l} + \underset{\sim}{s}$ in the usual notations.

Hence show that a finite rotation operator in Hilbert space is given by

$$R_{\underset{\sim}{n}}(\theta) = \exp(-i\theta \underset{\sim}{n}\cdot \underset{\sim}{J}/\hbar)$$

(b) Show that for a spin ½ rotation operator, it can be written as

$$R_{\underset{\sim}{n}}(\theta) = \exp(-i\theta \underset{\sim}{n}\cdot \underset{\sim}{s}/\hbar) = \cos\frac{\theta}{2} - i\underset{\sim}{n}\cdot\underset{\sim}{\sigma}\sin\frac{\theta}{2}$$

where $\underset{\sim}{\sigma}$ is the Pauli matrix with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note: The following formula can be used without proof,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

(c) An eigenstate of the σ_3 is given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Using the spin ½ rotation operator, show that an eigenstate of the spin operator $\underset{\sim}{n} \cdot \underset{\sim}{\sigma}$ is given by

$$\begin{pmatrix} e^{-i\phi/2} \cos\frac{\theta}{2} \\ e^{i\phi/2} \sin\frac{\theta}{2} \end{pmatrix}$$

where the unit vector $\underset{\sim}{n}$ is given by $\underset{\sim}{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

(d) Consider the decay of the particle Λ^0 into a proton p and a pion π^-,

$$\Lambda^0 \rightarrow p + \pi^-$$

The pion has spin zero, the proton and Λ^0 each has spin ½. The Λ^0 is polarized with its spin in the up-state when it disintegrates. What is the angular distribution of the disintegrations? That is, find the probability that the proton is detected at angle θ with respect to the 3[rd] coordinate axis, x_3-axis, given that (i) a is the probability amplitude of the disintegration with the proton being detected along the positive x_3-axis, namely, θ=0, when the spin state of the Λ^0 is in the **up**-state and (ii) b is the probability amplitude of the disintegration with the proton being detected along the positive x_3-axis, namely, θ=0, when the spin state of the Λ^0 is in the **down**-state .

4. **(a)** Consider two non-identical particles, with angular momentum $\underset{\sim}{J}_1$ and $\underset{\sim}{J}_2$ respectively and the associated quantum numbers $j_1 = j_2 = 1$. The Hamiltonian for the system is given by

$$H = \frac{E_1}{\hbar^2}(\underset{\sim}{J}_1 + \underset{\sim}{J}_2)\cdot \underset{\sim}{J}_2 + \frac{E_2}{\hbar^2}(J_{13} + J_{23})^2$$

where E_1 and E_2 are constant having the dimension of energy and $\underset{\sim}{J} = \underset{\sim}{J}_1 + \underset{\sim}{J}_2$.

Find the energy levels and the energy eigenstates. Determine the degeneracies of those states whose total angular momentum quantum number is $j = j_1 + j_2 = 2$.

Note: Use the Table of the Clebsch-Gordan coefficients.

(b) Consider two non-interacting particles, both of mass m, in a one-dimensional harmonic potential well

$$V(x) = \frac{1}{2} m\omega^2 x^2$$

The one-particle states are

$$\psi(X) = \frac{1}{\sqrt{n!\, 2^n}} \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} e^{-\frac{X^2}{2}} H_n(X), \qquad X = \sqrt{\frac{m\omega}{\hbar}}\, x,$$

with energies $E_n = (n + \frac{1}{2})\hbar\omega, \quad n = 0, 1, 2, \ldots$

Find the eigenfunctions and the corresponding energies of the ground state, the first and second excited states of the two-particle system. You should distinguish the two cases: (i) identical bosons, and (ii) identical fermions. Write down the ground state wavefunction of three identical fermions in the same potential well.

- The End -

CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A $\sqrt{\ }$ is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

1/2×1/2

m_1	m_2		
	J:	1	
	M:	+1	
+1/2	+1/2	1	
	J:	1	0
	M:	0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
	J:	1	
	M:	−1	
−1/2	−1/2	1	

1×1/2

m_1	m_2		
	J:	3/2	
	M:	+3/2	
+1	+1/2	1	
	J:	3/2	1/2
	M:	+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3
	J:	3/2	1/2
	M:	−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3
	J:	3/2	
	M:	−3/2	
−1	−1/2	1	

2×1/2

m_1	m_2		
	J:	5/2	
	M:	+5/2	
+2	+1/2	1	
	J:	5/2	3/2
	M:	+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5
	J:	5/2	3/2
	M:	+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5
	J:	5/2	3/2
	M:	−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5
	J:	5/2	3/2
	M:	−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5
	J:	5/2	
	M:	−5/2	
−2	−1/2	1	

3/2×1/2

m_1	m_2		
	J:	2	
	M:	+2	
+3/2	+1/2	1	
	J:	2	1
	M:	+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4
	J:	2	1
	M:	0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
	J:	2	1
	M:	−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4
	J:	2	
	M:	−2	
−3/2	−1/2	1	

2×1

m_1	m_2			
	J:	3		
	M:	+3		
+2	+1	1		
	J:	3	2	
	M:	+2	+2	
+2	0	1/3	2/3	
+1	+1	2/3	−1/3	
	J:	3	2	1
	M:	+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	6/15	−1/2	1/10
	J:	3	2	1
	M:	0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10
	J:	3	2	1
	M:	−1	−1	−1
0	−1	6/15	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5
	J:	3	2	
	M:	−2	−2	
−1	−1	2/3	1/3	
−2	0	1/3	−2/3	
	J:	3		
	M:	−3		
−2	−1	1		

1×1

m_1	m_2			
	J:	2		
	M:	+2		
+1	+1	1		
	J:	2	1	
	M:	+1	+1	
+1	0	1/2	1/2	
0	+1	1/2	−1/2	
	J:	2	1	0
	M:	0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3
	J:	2	1	
	M:	−1	−1	
0	−1	1/2	1/2	
−1	0	1/2	−1/2	
	J:	2		
	M:	−2		
−1	−1	1		

3/2×1

m_1	m_2			
	J:	5/2		
	M:	+5/2		
+3/2	+1	1		
	J:	5/2	3/2	
	M:	+3/2	+3/2	
+3/2	0	2/5	3/5	
+1/2	+1	3/5	−2/5	
	J:	5/2	3/2	1/2
	M:	+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6
	J:	5/2	3/2	1/2
	M:	−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2
	J:	5/2	3/2	
	M:	−3/2	−3/2	
−1/2	−1	3/5	2/5	
−3/2	0	2/5	−3/5	
	J:	5/2		
	M:	−5/2		
−3/2	−1	1		

3/2×3/2

m_1	m_2				
	J:	3			
	M:	+3			
+3/2	+3/2	1			
	J:	3	2		
	M:	+2	+2		
+3/2	+1/2	1/2	1/2		
+1/2	+3/2	1/2	−1/2		
	J:	3	2	1	
	M:	+1	+1	+1	
+3/2	−1/2	1/5	1/2	3/10	
+1/2	+1/2	3/5	0	−2/5	
−1/2	+3/2	1/5	−1/2	3/10	
	J:	3	2	1	0
	M:	0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4
	J:	3	2	1	
	M:	−1	−1	−1	
+1/2	−3/2	1/5	1/2	3/10	
−1/2	−1/2	3/5	0	−2/5	
−3/2	+1/2	1/5	−1/2	3/10	
	J:	3	2		
	M:	−2	−2		
−1/2	−3/2	1/2	1/2		
−3/2	−1/2	1/2	−1/2		
	J:	3			
	M:	−3			
−3/2	−3/2	1			

2×3/2

m_1	m_2				
	J:	7/2			
	M:	+7/2			
+2	+3/2	1			
	J:	7/2	5/2		
	M:	+5/2	+5/2		
+2	+1/2	3/7	4/7		
+1	+3/2	4/7	−3/7		
	J:	7/2	5/2	3/2	
	M:	+3/2	+3/2	+3/2	
+2	−1/2	1/7	16/35	2/5	
+1	+1/2	4/7	1/35	−2/5	
0	+3/2	2/7	−18/35	1/5	
	J:	7/2	5/2	3/2	1/2
	M:	+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10
	J:	7/2	5/2	3/2	1/2
	M:	−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5
	J:	7/2	5/2	3/2	
	M:	−3/2	−3/2	−3/2	
0	−3/2	2/7	18/35	1/5	
−1	−1/2	4/7	−1/35	−2/5	
−2	+1/2	1/7	−16/35	2/5	
	J:	7/2	5/2		
	M:	−5/2	−5/2		
−1	−3/2	4/7	3/7		
−2	−1/2	3/7	−4/7		
	J:	7/2			
	M:	−7/2			
−2	−3/2	1			

2×2

m_1	m_2					
	J:	4				
	M:	+4				
+2	+2	1				
	J:	4	3			
	M:	+3	+3			
+2	+1	1/2	1/2			
+1	+2	1/2	−1/2			
	J:	4	3	2		
	M:	+2	+2	+2		
+2	0	3/14	1/2	2/7		
+1	+1	4/7	0	−3/7		
0	+2	3/14	−1/2	2/7		
	J:	4	3	2	1	
	M:	+1	+1	+1	+1	
+2	−1	1/14	3/10	3/7	1/5	
+1	0	3/7	1/5	−1/14	−3/10	
0	+1	3/7	−1/5	−1/14	3/10	
−1	+2	1/14	−3/10	3/7	−1/5	
	J:	4	3	2	1	0
	M:	0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5
	J:	4	3	2	1	
	M:	−1	−1	−1	−1	
+1	−2	1/14	3/10	3/7	1/5	
0	−1	3/7	1/5	−1/14	−3/10	
−1	0	3/7	−1/5	−1/14	3/10	
−2	+1	1/14	−3/10	3/7	−1/5	
	J:	4	3	2		
	M:	−2	−2	−2		
0	−2	3/14	1/2	2/7		
−1	−1	4/7	0	−3/7		
−2	0	3/14	−1/2	2/7		
	J:	4	3			
	M:	−3	−3			
−1	−2	1/2	1/2			
−2	−1	1/2	−1/2			
	J:	4				
	M:	−4				
−2	−2	1				

$$d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2} \qquad d^1_{0,0} = \cos\theta$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta+1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 20.1: Sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The signs and numbers in the current tables have been calculated by computer programs written independently by Cohen and at LBL.