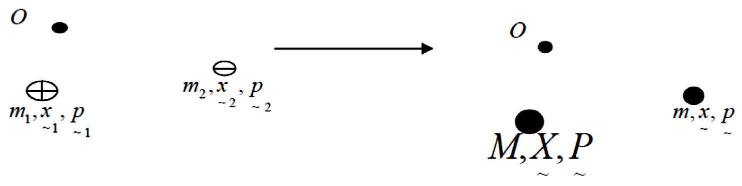
Question 1(a)

The Hamiltonian for the hydrogen atom, one proton and one electron with masses m_p and m_e,

$$H = \frac{p_p^2}{2m_p} + \frac{p_e^2}{2m_e} + V(\vec{x}_p - \vec{x}_e)$$

We transform them into 2 equivalent particles M and m:



We let

$$M = m_p + m_e \approx m_p, \qquad m = \frac{m_p m_e}{m_p + m_e} \approx m_e$$

Then we get

$$M\vec{X} = m_p \vec{x}_p + m_e \vec{x}_e, \qquad \vec{x} = \vec{x}_p - \vec{x}_e$$

The momentum,

$$\vec{P} = M\dot{\vec{X}}, \qquad \vec{p} = m\dot{\vec{x}}$$

Rearranging, we get

$$\vec{p}_p = \frac{m}{m_e}\vec{P} + \vec{p} \approx \vec{P} + \vec{p}, \qquad \vec{p}_e = \frac{m}{m_p}\vec{P} - \vec{p} \approx -\vec{p}$$

Using the above relations, the Hamiltonian now becomes

$$\begin{aligned} H &= \frac{1}{2m_p}\left(\frac{m}{m_e}\vec{P} + \vec{p}\right)^2 + \frac{1}{2m_e}\left(\frac{m}{m_p}\vec{P} - \vec{p}\right)^2 + V(\vec{x}) \\ &= \frac{1}{2m_p}\left[\frac{m^2}{m_e^2}P^2 + \frac{m}{m_e}(\vec{P}\cdot\vec{p} + \vec{p}\cdot\vec{P}) + p^2\right] + \frac{1}{2m_e}\left[\frac{m^2}{m_p^2}P^2 + \frac{m}{m_e}(\vec{P}\cdot\vec{p} - \vec{p}\cdot\vec{P}) + p^2\right] + V(\vec{x}) \\ &= \frac{P^2}{2(m_p + m_e)} + \frac{p^2}{2}\left(\frac{m_p + m_e}{m_p m_e}\right) + V(\vec{x}) \\ &= \frac{P^2}{2M} + \frac{p^2}{2m} + V(\vec{x}) \end{aligned}$$

Where M, P describes the motion of the center of mass, and m, p describes the relative motion of the electron and the proton. [shown]

Question 1(b)

We know that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}, \qquad \frac{\partial \theta}{\partial x_i} = \frac{1}{r\sin\theta}\left(\frac{x_i}{r}\cos\theta - \delta_{i3}\right), \qquad \frac{\partial \phi}{\partial x_i} = \frac{1}{r\sin\theta}(\delta_{i2}\cos\phi - \delta_{i1}\sin\phi)$$

Therefore we have

$$\begin{aligned} l_3 &= x_1 p_2 - x_2 p_1 \\ &= \frac{\hbar}{i}\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) \\ &= \frac{\hbar}{i}\left[x_1\left(\frac{\partial r}{\partial x_2}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x_2}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x_2}\frac{\partial}{\partial \phi}\right) - x_2\left(\frac{\partial r}{\partial x_1}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x_1}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x_1}\frac{\partial}{\partial \phi}\right)\right] \end{aligned}$$

$$= \frac{\hbar}{i}\left[x_1\left(\frac{x_2}{r}\frac{\partial}{\partial r} + \frac{x_2\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) - x_2\left(\frac{x_1}{r}\frac{\partial}{\partial r} + \frac{x_1\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right]$$
$$= \frac{\hbar}{i}\left(\frac{x_1\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{x_2\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$
$$= \frac{\hbar}{i}\left(\cos^2\phi\frac{\partial}{\partial\phi} + \sin^2\phi\frac{\partial}{\partial\phi}\right)$$
$$= \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

Let ψ be an arbitrary function, hence we have

$$\left[\phi, -i\hbar\frac{\partial}{\partial\phi}\right]\psi = -i\hbar\phi\frac{\partial\psi}{\partial\phi} + i\hbar\frac{\partial}{\partial\phi}(\phi\psi) = -i\hbar\phi\frac{\partial\psi}{\partial\phi} + i\hbar\psi + i\hbar\phi\frac{\partial\psi}{\partial\phi} = i\hbar\psi$$

$$\therefore \left[\phi, -i\hbar\frac{\partial}{\partial\phi}\right] = i\hbar \quad \text{[shown]}$$

Suppose $[\phi, l_3] = i\hbar$, then $\langle l,m|[\phi,l_3]|l,m\rangle = i\hbar\langle l,m|l,m\rangle$

$LHS = \langle l,m|[\phi,l_3]|l,m\rangle = \langle l,m|(\phi l_3 - l_3\phi)|l,m\rangle = \langle l,m|(\phi m\hbar - m\hbar\phi)|l,m\rangle = 0$

$RHS = i\hbar\langle l,m|l,m\rangle \neq 0$

$$\therefore [\phi, l_3] \neq i\hbar$$

In coordinate representation,

$$l_+ = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right), \qquad l_3 = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

With $l_3 Y_l^l(\theta,\phi) = l\hbar Y_l^l(\theta,\phi)$, we have

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}Y_l^l(\theta,\phi) = l\hbar Y_l^l(\theta,\phi), \qquad \Rightarrow \quad Y_l^l(\theta,\phi) = f(\theta)e^{il\phi}$$

Where $f(\theta)$ is an arbitrary function of θ. Substitute into the representation of l_+,

$$l_+ Y_l^l(\theta,\phi) = 0$$
$$\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)f(\theta)e^{il\phi} = 0$$
$$\left(\frac{\partial}{\partial\theta} - l\cot\theta\right)f(\theta) = 0$$
$$\frac{\partial f}{\partial\theta} = lf\cot\theta$$
$$\int\frac{1}{f}df = \int l\cot\theta\,d\theta$$
$$\ln f = l\ln(\sin\theta) + c$$
$$f = N\sin^l\theta$$

$$\therefore Y_l^l(\theta,\phi) = N\sin^l\theta\, e^{il\phi} \quad \text{[shown]}$$

Question 2(a)

$$H = \frac{mv^2}{2} + V = \frac{(\vec{p} - q\vec{A})^2}{2m} + qA_0$$

Using the formula given, we have

$$\begin{aligned}
\frac{d}{dt}\langle x_i \rangle &= \frac{i}{\hbar}\langle [H, x_i] \rangle + \underbrace{\langle \frac{\partial x_i}{\partial t} \rangle}_{=0} \\
&= \frac{i}{\hbar}\langle \left[\frac{(\vec{p} - q\vec{A})^2}{2m} + qA_0, x_i \right] \rangle \\
&= \frac{i}{2m\hbar}\langle [(p - qA)_j (p - qA)_j, x_i] \rangle \\
&= \frac{i}{2m\hbar}\langle (p - qA)_j [(p - qA)_j, x_i] + [(p - qA)_j, x_i](p - qA)_j \rangle \\
&= \frac{i}{2m\hbar}\langle (p - qA)_j (-i\hbar\delta_{ij}) + (-i\hbar\delta_{ij})(p - qA)_j \rangle \\
&= \frac{i}{2m\hbar}\langle -2i\hbar (p - qA)_j \rangle \\
&= \frac{1}{m}\langle (p - qA)_j \rangle
\end{aligned}$$

$$\therefore \frac{d}{dt}\langle \vec{x} \rangle = \frac{1}{m}\langle (\vec{p} - q\vec{A}) \rangle$$

Question 2(b)(i)

We know that $\vec{B} = \vec{\nabla} \times \vec{A}$. Then also,

$$\vec{A} = \frac{B_0}{2}(x_1\hat{\jmath} - x_2\hat{\imath}) = \frac{1}{2}\vec{B} \times \vec{x}, \qquad \vec{x} = x_1\hat{\imath} + x_2\hat{\jmath} + x_3\hat{k}, \qquad \vec{B} = B_0\hat{k}$$

We substitute $\vec{A}$ into the formula,

$$\begin{aligned}
\vec{\nabla} \times \vec{A} &= \vec{\nabla} \times \left(\frac{1}{2}\vec{B} \times \vec{x} \right) \\
&= \frac{1}{2}\left[(\vec{x} \cdot \vec{\nabla})\vec{B} - (\vec{B} \cdot \vec{\nabla})\vec{x} + \vec{B}(\vec{\nabla} \cdot \vec{x}) - \vec{x}(\vec{\nabla} \cdot \vec{B}) \right] \\
&= \frac{1}{2}\left[0 - B_0\hat{k} + 3B_0\hat{k} - 0 \right] \\
&= B_0\hat{k} \\
&= \vec{B}
\end{aligned}$$

Next, we know that $\vec{E} = -2Cx_3\hat{k}$. Substituting A_0 into the formula

$$-\vec{\nabla}A_0 + \underbrace{\frac{\partial \vec{A}}{\partial t}}_{=0} = -\vec{\nabla}(Cx_3^2) = -2Cx_3\hat{k}$$

$$\therefore \vec{A} = \frac{B_0}{2}(x_1\hat{\jmath} - x_2\hat{\imath}), \qquad A_0 = Cx_3^2 \quad \text{[shown]}$$

Question 2(b)(ii)
The Hamiltonian now becomes

$$\begin{aligned}
H &= \frac{\left(\vec{p} - q\vec{A}\right)^2}{2m} + qA_0 \\
&= \frac{\left[\vec{p} - q\frac{B_0}{2}(x_1\hat{j} - x_2\hat{\imath})\right]^2}{2m} + qCx_3^2 \\
&= \frac{1}{2m}\left[p^2 + \frac{q^2B_0^2}{4}(x_1^2 + x_2^2) - qB_0l_3\right] + qCx_3^2 \\
&= \left[\frac{1}{2m}(p_1^2 + p_2^2) + \frac{q^2B_0^2}{8m}(x_1^2 + x_2^2)\right] + \left[\frac{p_3^2}{2m} + qCx_3^2\right] - \frac{qB_0l_3}{2m} \\
&= \left[\frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}m\underbrace{\left(\frac{q^2B_0^2}{4m^2}\right)}_{=\omega_1^2}(x_1^2 + x_2^2)\right] + \left[\frac{p_3^2}{2m} + \frac{1}{2}m\underbrace{\left(\frac{2qC}{m}\right)}_{=\omega_2^2}x_3^2\right] - \frac{qB_0l_3}{2m}
\end{aligned}$$

Which is a combination of 2 harmonic oscillators. We then find the energies to be

$$E(n_1, n_2) = \left(n_1 + \frac{1}{2}\right)\hbar\omega_1 + \left(n_2 + \frac{1}{2}\right)\hbar\omega_2 \quad \text{[shown]}$$

When $\vec{E} = 0$, we get the energy $E = \left(n_1 + \frac{1}{2}\right)\hbar\omega_1$, which is the Landau level of the particle.

Question 3(a)

$$\mathcal{R}_{\vec{u}}(\varepsilon) = 1 + \varepsilon\vec{u}\times + \frac{1}{2}\varepsilon^2\vec{u}\times\vec{u}\times + O(\varepsilon^3)$$

$$\begin{aligned}
\mathcal{R}_{\vec{v}}\mathcal{R}_{\vec{u}} &= \left(1 + \varepsilon\vec{v}\times + \frac{1}{2}\varepsilon^2\vec{v}\times\vec{v}\times + O(\varepsilon^3)\right)\left(1 + \varepsilon\vec{u}\times + \frac{1}{2}\varepsilon^2\vec{u}\times\vec{u}\times + O(\varepsilon^3)\right) \\
&= 1 + \varepsilon(\vec{u} + \vec{v})\times + \varepsilon^2\vec{v}\times\vec{u}\times + \frac{1}{2}\varepsilon^2\vec{u}\times\vec{u}\times + \frac{1}{2}\varepsilon^2\vec{v}\times\vec{v}\times + O(\varepsilon^3)
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{\vec{u}}^{-1}\mathcal{R}_{\vec{v}}\mathcal{R}_{\vec{u}} &= \left(1 - \varepsilon\vec{u}\times + \frac{1}{2}\varepsilon^2\vec{u}\times\vec{u}\times + O(\varepsilon^3)\right)\left[1 + \varepsilon(\vec{u} + \vec{v})\times + \varepsilon^2\vec{v}\times\vec{u}\times + \frac{1}{2}\varepsilon^2\vec{u}\times\vec{u}\times + \frac{1}{2}\varepsilon^2\vec{v}\times\vec{v}\times + O(\varepsilon^3)\right] \\
&= 1 + \varepsilon\vec{v}\times + \varepsilon^2\vec{v}\times\vec{u}\times - \varepsilon^2\vec{u}\times\vec{v}\times + \frac{1}{2}\varepsilon^2\vec{v}\times\vec{v}\times + O(\varepsilon^3)
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{\vec{v}}^{-1}\mathcal{R}_{\vec{u}}^{-1}\mathcal{R}_{\vec{v}}\mathcal{R}_{\vec{u}} &= \left(1 - \varepsilon\vec{v}\times + \frac{1}{2}\varepsilon^2\vec{v}\times\vec{v}\times + O(\varepsilon^3)\right)\left(1 + \varepsilon\vec{v}\times + \varepsilon^2\vec{v}\times\vec{u}\times - \varepsilon^2\vec{u}\times\vec{v}\times + \frac{1}{2}\varepsilon^2\vec{v}\times\vec{v}\times + O(\varepsilon^3)\right) \\
&= 1 + \varepsilon^2\vec{v}\times\vec{u}\times - \varepsilon^2\vec{u}\times\vec{v}\times + O(\varepsilon^3)
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{\vec{v}}^{-1}\mathcal{R}_{\vec{u}}^{-1}\mathcal{R}_{\vec{v}}\mathcal{R}_{\vec{u}}\vec{x} &= \vec{x} + \varepsilon^2\vec{v}\times(\vec{u}\times\vec{x}) - \varepsilon^2\vec{u}\times(\vec{v}\times\vec{x}) + O(\varepsilon^3) \\
&= \vec{x} + \varepsilon^2[\vec{u}(\vec{v}\cdot\vec{x}) - \vec{v}(\vec{u}\cdot\vec{x})] + O(\varepsilon^3) \\
&= \vec{x} + \varepsilon^2\vec{x}\times(\vec{u}\times\vec{v}) + O(\varepsilon^3) \\
&= \vec{x} - \varepsilon^2\vec{w}\times\vec{x} + O(\varepsilon^3)
\end{aligned}$$

But we know that $\mathcal{R}_{\vec{w}}(-\varepsilon^2) = \vec{x} - \varepsilon^2\vec{w}\times\vec{x} + O(\varepsilon^4)$. $\therefore \mathcal{R}_{\vec{v}}^{-1}\mathcal{R}_{\vec{u}}^{-1}\mathcal{R}_{\vec{v}}\mathcal{R}_{\vec{u}}$ differs from $\mathcal{R}_{\vec{w}}(-\varepsilon^2)$ only by terms of higher order than ε^2. [shown]

Question 3(b)

$$J_1^2|j_1,m_1\rangle = \frac{3}{4}\hbar^2|j_1,m_1\rangle = j_1(j_1+1)\hbar^2|j_1,m_1\rangle$$
$$j_1^2 + j_1 - \frac{3}{4} = 0 \quad \Rightarrow \quad j_1 = -\frac{3}{2},\frac{1}{2}$$

$$J_2^2|j_2,m_2\rangle = 6\hbar^2|j_2,m_2\rangle = j_2(j_2+1)\hbar^2|j_2,m_2\rangle$$
$$j_2^2 + j_2 - 6 = 0 \quad \Rightarrow \quad j_2 = -3,2$$

So we have $j_1 = \frac{1}{2}, j_2 = 2$. Using the Clebsch-Gordan Coefficient table, and looking for $|j,m\rangle \Rightarrow |j_2,j_1,m_2,m_1\rangle$, we get a set of simultaneous normalized eigenvectors,

$$\left|\frac{5}{2},\frac{5}{2}\right\rangle = \left|2,\frac{1}{2},2,\frac{1}{2}\right\rangle$$
$$\left|\frac{5}{2},\frac{3}{2}\right\rangle = \frac{1}{\sqrt{5}}\left|2,\frac{1}{2},2,-\frac{1}{2}\right\rangle + \frac{2}{\sqrt{5}}\left|2,\frac{1}{2},1,\frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2},\frac{3}{2}\right\rangle = \frac{2}{\sqrt{5}}\left|2,\frac{1}{2},2,-\frac{1}{2}\right\rangle - \frac{1}{\sqrt{5}}\left|2,\frac{1}{2},1,\frac{1}{2}\right\rangle$$
$$\left|\frac{5}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{5}}\left|2,\frac{1}{2},1,-\frac{1}{2}\right\rangle + \sqrt{\frac{3}{5}}\left|2,\frac{1}{2},0,\frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{3}{5}}\left|2,\frac{1}{2},1,-\frac{1}{2}\right\rangle - \sqrt{\frac{2}{5}}\left|2,\frac{1}{2},0,\frac{1}{2}\right\rangle$$
$$\left|\frac{5}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{3}{5}}\left|2,\frac{1}{2},0,-\frac{1}{2}\right\rangle + \sqrt{\frac{2}{5}}\left|2,\frac{1}{2},-1,\frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{5}}\left|2,\frac{1}{2},0,-\frac{1}{2}\right\rangle - \sqrt{\frac{3}{5}}\left|2,\frac{1}{2},-1,\frac{1}{2}\right\rangle$$
$$\left|\frac{5}{2},-\frac{3}{2}\right\rangle = \frac{2}{\sqrt{5}}\left|2,\frac{1}{2},-1,-\frac{1}{2}\right\rangle + \frac{1}{\sqrt{5}}\left|2,\frac{1}{2},-2,\frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \frac{1}{\sqrt{5}}\left|2,\frac{1}{2},-1,-\frac{1}{2}\right\rangle - \frac{2}{\sqrt{5}}\left|2,\frac{1}{2},-2,\frac{1}{2}\right\rangle$$
$$\left|\frac{5}{2},-\frac{5}{2}\right\rangle = \left|2,\frac{1}{2},-2,-\frac{1}{2}\right\rangle$$

Question 4(a)

Let c_m be the number of states $|j, m\rangle$ for a fixed $m = m_1 + m_2$, and d_m be the number of states $|j, m\rangle$ for a fixed m and j. With $|m| \leq j$, and assume $m \geq 0$,

$$c_m = \sum_{m \geq 0} d_m = d_m + d_{m+1} + \cdots$$
$$c_{m+1} = d_{m+1} + d_{m+2} + \cdots$$
$$c_m - c_{m+1} = d_m$$

$\boldsymbol{m}$	$\boldsymbol{m_1}$	$\boldsymbol{m_2}$	$\boldsymbol{c_m}$
$j_1 + j_2$	j_1	j_2	1
$j_1 + j_2 - 1$	$j_1 - 1$ j_1	j_2 $j_2 - 1$	2
$j_1 + j_2 - 2$	$j_1 - 2$ $j_1 - 1$ j_1	j_2 $j_2 - 1$ $j_2 - 2$	3
…	…	…	…
$j_1 + j_2 - n$			$n + 1$

So if you know c_m, we can get d_m. As in the case of a simple harmonic oscillator, we know that $d_j = 1$ if j is allowed, and $d_j = 0$ if j is not allowed. The value of n to terminate the counting is $-2j_2$ if $j_2 < j_1$, or $-2j_1$ if $j_1 < j_2$.

$$\therefore j = j_1 + j_2, j_1 + j_2 - 1, j_1 + j_2 - 2, \ldots, |j_1 - j_2| \quad \text{[shown]}$$

Question 4(b)

$$E = \frac{\hbar^2 \pi^2}{2mb^2}(k^2 + l^2 + m^2 + n^2)$$

<u>(i) For distinguishable identical particles,</u>

$$\psi = \frac{4}{b^2} \sin\left(\frac{k\pi x_1}{b}\right) \sin\left(\frac{l\pi x_2}{b}\right) \sin\left(\frac{m\pi x_3}{b}\right) \sin\left(\frac{n\pi x_4}{b}\right)$$

Ground state, $\quad E = \dfrac{2\hbar^2\pi^2}{mb^2}$

$$\psi = \frac{4}{b^2} \sin\left(\frac{\pi x_1}{b}\right) \sin\left(\frac{\pi x_2}{b}\right) \sin\left(\frac{\pi x_3}{b}\right) \sin\left(\frac{\pi x_4}{b}\right)$$

1st excited state, $\quad E = \dfrac{7\hbar^2\pi^2}{2mb^2}$

$$\psi = \frac{4}{b^2} \sin\left(\frac{\pi x_1}{b}\right) \sin\left(\frac{\pi x_2}{b}\right) \sin\left(\frac{\pi x_3}{b}\right) \sin\left(\frac{2\pi x_4}{b}\right)$$
$$\psi = \frac{4}{b^2} \sin\left(\frac{\pi x_1}{b}\right) \sin\left(\frac{\pi x_2}{b}\right) \sin\left(\frac{2\pi x_3}{b}\right) \sin\left(\frac{\pi x_4}{b}\right)$$
$$\psi = \frac{4}{b^2} \sin\left(\frac{\pi x_1}{b}\right) \sin\left(\frac{2\pi x_2}{b}\right) \sin\left(\frac{\pi x_3}{b}\right) \sin\left(\frac{\pi x_4}{b}\right)$$
$$\psi = \frac{4}{b^2} \sin\left(\frac{2\pi x_1}{b}\right) \sin\left(\frac{\pi x_2}{b}\right) \sin\left(\frac{\pi x_3}{b}\right) \sin\left(\frac{\pi x_4}{b}\right)$$

(ii) For identical bosons,

$$\begin{aligned}\psi = \frac{1}{b^2\sqrt{6}}\Big[&\sin\left(\frac{k\pi x_1}{b}\right)\sin\left(\frac{l\pi x_2}{b}\right)\sin\left(\frac{m\pi x_3}{b}\right)\sin\left(\frac{n\pi x_4}{b}\right)+\sin\left(\frac{k\pi x_2}{b}\right)\sin\left(\frac{l\pi x_3}{b}\right)\sin\left(\frac{m\pi x_4}{b}\right)\sin\left(\frac{n\pi x_1}{b}\right)\\
&+\sin\left(\frac{k\pi x_3}{b}\right)\sin\left(\frac{l\pi x_4}{b}\right)\sin\left(\frac{m\pi x_1}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)+\sin\left(\frac{k\pi x_4}{b}\right)\sin\left(\frac{l\pi x_1}{b}\right)\sin\left(\frac{m\pi x_2}{b}\right)\sin\left(\frac{n\pi x_3}{b}\right)\\
&+\sin\left(\frac{k\pi x_2}{b}\right)\sin\left(\frac{l\pi x_1}{b}\right)\sin\left(\frac{m\pi x_3}{b}\right)\sin\left(\frac{n\pi x_4}{b}\right)+\sin\left(\frac{k\pi x_1}{b}\right)\sin\left(\frac{l\pi x_3}{b}\right)\sin\left(\frac{m\pi x_4}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)\\
&+\sin\left(\frac{k\pi x_3}{b}\right)\sin\left(\frac{l\pi x_4}{b}\right)\sin\left(\frac{m\pi x_2}{b}\right)\sin\left(\frac{n\pi x_1}{b}\right)+\sin\left(\frac{k\pi x_4}{b}\right)\sin\left(\frac{l\pi x_2}{b}\right)\sin\left(\frac{m\pi x_1}{b}\right)\sin\left(\frac{n\pi x_3}{b}\right)\\
&+\sin\left(\frac{k\pi x_1}{b}\right)\sin\left(\frac{l\pi x_3}{b}\right)\sin\left(\frac{m\pi x_2}{b}\right)\sin\left(\frac{n\pi x_4}{b}\right)+\sin\left(\frac{k\pi x_3}{b}\right)\sin\left(\frac{l\pi x_2}{b}\right)\sin\left(\frac{m\pi x_4}{b}\right)\sin\left(\frac{n\pi x_1}{b}\right)\\
&+\sin\left(\frac{k\pi x_2}{b}\right)\sin\left(\frac{l\pi x_4}{b}\right)\sin\left(\frac{m\pi x_1}{b}\right)\sin\left(\frac{n\pi x_3}{b}\right)+\sin\left(\frac{k\pi x_4}{b}\right)\sin\left(\frac{l\pi x_1}{b}\right)\sin\left(\frac{m\pi x_3}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)\\
&+\sin\left(\frac{k\pi x_1}{b}\right)\sin\left(\frac{l\pi x_3}{b}\right)\sin\left(\frac{m\pi x_4}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)+\sin\left(\frac{k\pi x_3}{b}\right)\sin\left(\frac{l\pi x_4}{b}\right)\sin\left(\frac{m\pi x_2}{b}\right)\sin\left(\frac{n\pi x_1}{b}\right)\\
&+\sin\left(\frac{k\pi x_4}{b}\right)\sin\left(\frac{l\pi x_2}{b}\right)\sin\left(\frac{m\pi x_1}{b}\right)\sin\left(\frac{n\pi x_3}{b}\right)+\sin\left(\frac{k\pi x_2}{b}\right)\sin\left(\frac{l\pi x_1}{b}\right)\sin\left(\frac{m\pi x_3}{b}\right)\sin\left(\frac{n\pi x_4}{b}\right)\\
&+\sin\left(\frac{k\pi x_1}{b}\right)\sin\left(\frac{l\pi x_4}{b}\right)\sin\left(\frac{m\pi x_3}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)+\sin\left(\frac{k\pi x_4}{b}\right)\sin\left(\frac{l\pi x_3}{b}\right)\sin\left(\frac{m\pi x_2}{b}\right)\sin\left(\frac{n\pi x_1}{b}\right)\\
&+\sin\left(\frac{k\pi x_3}{b}\right)\sin\left(\frac{l\pi x_2}{b}\right)\sin\left(\frac{m\pi x_1}{b}\right)\sin\left(\frac{n\pi x_4}{b}\right)+\sin\left(\frac{k\pi x_2}{b}\right)\sin\left(\frac{l\pi x_1}{b}\right)\sin\left(\frac{m\pi x_4}{b}\right)\sin\left(\frac{n\pi x_3}{b}\right)\\
&+\sin\left(\frac{k\pi x_1}{b}\right)\sin\left(\frac{l\pi x_4}{b}\right)\sin\left(\frac{m\pi x_2}{b}\right)\sin\left(\frac{n\pi x_3}{b}\right)+\sin\left(\frac{k\pi x_4}{b}\right)\sin\left(\frac{l\pi x_2}{b}\right)\sin\left(\frac{m\pi x_3}{b}\right)\sin\left(\frac{n\pi x_1}{b}\right)\\
&+\sin\left(\frac{k\pi x_2}{b}\right)\sin\left(\frac{l\pi x_3}{b}\right)\sin\left(\frac{m\pi x_1}{b}\right)\sin\left(\frac{n\pi x_4}{b}\right)+\sin\left(\frac{k\pi x_3}{b}\right)\sin\left(\frac{l\pi x_1}{b}\right)\sin\left(\frac{m\pi x_4}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)\Big]\end{aligned}$$

Ground state, $E = \dfrac{2\hbar^2\pi^2}{mb^2}$

$$\psi = \frac{24}{b^2\sqrt{6}}\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{\pi x_4}{b}\right)$$

1st excited state, $E = \dfrac{7\hbar^2\pi^2}{2mb^2}$

$$\begin{aligned}\psi = \frac{\sqrt{6}}{b^2}\Bigg[&\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)+\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{\pi x_1}{b}\right)\\
&+\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{\pi x_2}{b}\right)+\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{\pi x_3}{b}\right)\Bigg]\end{aligned}$$

(iii) For identical fermions,

$$\begin{aligned}\psi = \frac{1}{b^2\sqrt{6}}\Big[&\sin\left(\frac{k\pi x_1}{b}\right)\sin\left(\frac{l\pi x_2}{b}\right)\sin\left(\frac{m\pi x_3}{b}\right)\sin\left(\frac{n\pi x_4}{b}\right)+\sin\left(\frac{k\pi x_2}{b}\right)\sin\left(\frac{l\pi x_3}{b}\right)\sin\left(\frac{m\pi x_4}{b}\right)\sin\left(\frac{n\pi x_1}{b}\right)\\
&+\sin\left(\frac{k\pi x_3}{b}\right)\sin\left(\frac{l\pi x_4}{b}\right)\sin\left(\frac{m\pi x_1}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)+\sin\left(\frac{k\pi x_4}{b}\right)\sin\left(\frac{l\pi x_1}{b}\right)\sin\left(\frac{m\pi x_2}{b}\right)\sin\left(\frac{n\pi x_3}{b}\right)\\
&-\sin\left(\frac{k\pi x_2}{b}\right)\sin\left(\frac{l\pi x_1}{b}\right)\sin\left(\frac{m\pi x_3}{b}\right)\sin\left(\frac{n\pi x_4}{b}\right)-\sin\left(\frac{k\pi x_1}{b}\right)\sin\left(\frac{l\pi x_3}{b}\right)\sin\left(\frac{m\pi x_4}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)\\
&-\sin\left(\frac{k\pi x_3}{b}\right)\sin\left(\frac{l\pi x_4}{b}\right)\sin\left(\frac{m\pi x_2}{b}\right)\sin\left(\frac{n\pi x_1}{b}\right)-\sin\left(\frac{k\pi x_4}{b}\right)\sin\left(\frac{l\pi x_2}{b}\right)\sin\left(\frac{m\pi x_1}{b}\right)\sin\left(\frac{n\pi x_3}{b}\right)\\
&+\sin\left(\frac{k\pi x_1}{b}\right)\sin\left(\frac{l\pi x_3}{b}\right)\sin\left(\frac{m\pi x_2}{b}\right)\sin\left(\frac{n\pi x_4}{b}\right)+\sin\left(\frac{k\pi x_3}{b}\right)\sin\left(\frac{l\pi x_2}{b}\right)\sin\left(\frac{m\pi x_4}{b}\right)\sin\left(\frac{n\pi x_1}{b}\right)\\
&+\sin\left(\frac{k\pi x_2}{b}\right)\sin\left(\frac{l\pi x_4}{b}\right)\sin\left(\frac{m\pi x_1}{b}\right)\sin\left(\frac{n\pi x_3}{b}\right)+\sin\left(\frac{k\pi x_4}{b}\right)\sin\left(\frac{l\pi x_1}{b}\right)\sin\left(\frac{m\pi x_3}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)\\
&-\sin\left(\frac{k\pi x_1}{b}\right)\sin\left(\frac{l\pi x_3}{b}\right)\sin\left(\frac{m\pi x_4}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)-\sin\left(\frac{k\pi x_3}{b}\right)\sin\left(\frac{l\pi x_4}{b}\right)\sin\left(\frac{m\pi x_2}{b}\right)\sin\left(\frac{n\pi x_1}{b}\right)\\
&-\sin\left(\frac{k\pi x_4}{b}\right)\sin\left(\frac{l\pi x_2}{b}\right)\sin\left(\frac{m\pi x_1}{b}\right)\sin\left(\frac{n\pi x_3}{b}\right)-\sin\left(\frac{k\pi x_2}{b}\right)\sin\left(\frac{l\pi x_1}{b}\right)\sin\left(\frac{m\pi x_3}{b}\right)\sin\left(\frac{n\pi x_4}{b}\right)\\
&+\sin\left(\frac{k\pi x_1}{b}\right)\sin\left(\frac{l\pi x_4}{b}\right)\sin\left(\frac{m\pi x_3}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)+\sin\left(\frac{k\pi x_4}{b}\right)\sin\left(\frac{l\pi x_3}{b}\right)\sin\left(\frac{m\pi x_2}{b}\right)\sin\left(\frac{n\pi x_1}{b}\right)\\
&+\sin\left(\frac{k\pi x_3}{b}\right)\sin\left(\frac{l\pi x_2}{b}\right)\sin\left(\frac{m\pi x_1}{b}\right)\sin\left(\frac{n\pi x_4}{b}\right)+\sin\left(\frac{k\pi x_2}{b}\right)\sin\left(\frac{l\pi x_1}{b}\right)\sin\left(\frac{m\pi x_4}{b}\right)\sin\left(\frac{n\pi x_3}{b}\right)\\
&-\sin\left(\frac{k\pi x_1}{b}\right)\sin\left(\frac{l\pi x_4}{b}\right)\sin\left(\frac{m\pi x_2}{b}\right)\sin\left(\frac{n\pi x_3}{b}\right)-\sin\left(\frac{k\pi x_4}{b}\right)\sin\left(\frac{l\pi x_2}{b}\right)\sin\left(\frac{m\pi x_3}{b}\right)\sin\left(\frac{n\pi x_1}{b}\right)\\
&-\sin\left(\frac{k\pi x_2}{b}\right)\sin\left(\frac{l\pi x_3}{b}\right)\sin\left(\frac{m\pi x_1}{b}\right)\sin\left(\frac{n\pi x_4}{b}\right)-\sin\left(\frac{k\pi x_3}{b}\right)\sin\left(\frac{l\pi x_1}{b}\right)\sin\left(\frac{m\pi x_4}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)\Big]\end{aligned}$$

Ground state, $E = \dfrac{15\hbar^2\pi^2}{mb^2}$

$$
\begin{aligned}
\psi = \frac{1}{b^2\sqrt{6}}\Big[&\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{2\pi x_2}{b}\right)\sin\left(\frac{3\pi x_3}{b}\right)\sin\left(\frac{4\pi x_4}{b}\right)+\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{2\pi x_3}{b}\right)\sin\left(\frac{3\pi x_4}{b}\right)\sin\left(\frac{4\pi x_1}{b}\right)\\
&+\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{3\pi x_1}{b}\right)\sin\left(\frac{4\pi x_2}{b}\right)+\sin\left(\frac{\pi x_4}{b}\right)\sin\left(\frac{2\pi x_1}{b}\right)\sin\left(\frac{3\pi x_2}{b}\right)\sin\left(\frac{4\pi x_3}{b}\right)\\
&-\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{2\pi x_1}{b}\right)\sin\left(\frac{3\pi x_3}{b}\right)\sin\left(\frac{4\pi x_4}{b}\right)-\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{2\pi x_3}{b}\right)\sin\left(\frac{3\pi x_4}{b}\right)\sin\left(\frac{4\pi x_2}{b}\right)\\
&-\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{3\pi x_2}{b}\right)\sin\left(\frac{4\pi x_1}{b}\right)-\sin\left(\frac{\pi x_4}{b}\right)\sin\left(\frac{2\pi x_2}{b}\right)\sin\left(\frac{3\pi x_1}{b}\right)\sin\left(\frac{4\pi x_3}{b}\right)\\
&+\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{2\pi x_3}{b}\right)\sin\left(\frac{3\pi x_2}{b}\right)\sin\left(\frac{4\pi x_4}{b}\right)+\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_2}{b}\right)\sin\left(\frac{3\pi x_4}{b}\right)\sin\left(\frac{4\pi x_1}{b}\right)\\
&+\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{3\pi x_1}{b}\right)\sin\left(\frac{4\pi x_3}{b}\right)+\sin\left(\frac{\pi x_4}{b}\right)\sin\left(\frac{2\pi x_1}{b}\right)\sin\left(\frac{3\pi x_3}{b}\right)\sin\left(\frac{4\pi x_2}{b}\right)\\
&-\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{2\pi x_3}{b}\right)\sin\left(\frac{3\pi x_4}{b}\right)\sin\left(\frac{4\pi x_2}{b}\right)-\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{3\pi x_2}{b}\right)\sin\left(\frac{4\pi x_1}{b}\right)\\
&-\sin\left(\frac{\pi x_4}{b}\right)\sin\left(\frac{2\pi x_2}{b}\right)\sin\left(\frac{3\pi x_1}{b}\right)\sin\left(\frac{4\pi x_3}{b}\right)-\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{2\pi x_1}{b}\right)\sin\left(\frac{3\pi x_3}{b}\right)\sin\left(\frac{4\pi x_4}{b}\right)\\
&+\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{3\pi x_3}{b}\right)\sin\left(\frac{4\pi x_2}{b}\right)+\sin\left(\frac{\pi x_4}{b}\right)\sin\left(\frac{2\pi x_3}{b}\right)\sin\left(\frac{3\pi x_2}{b}\right)\sin\left(\frac{4\pi x_1}{b}\right)\\
&+\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_2}{b}\right)\sin\left(\frac{3\pi x_1}{b}\right)\sin\left(\frac{4\pi x_4}{b}\right)+\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{2\pi x_1}{b}\right)\sin\left(\frac{3\pi x_4}{b}\right)\sin\left(\frac{4\pi x_3}{b}\right)\\
&-\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{3\pi x_2}{b}\right)\sin\left(\frac{4\pi x_3}{b}\right)-\sin\left(\frac{\pi x_4}{b}\right)\sin\left(\frac{2\pi x_2}{b}\right)\sin\left(\frac{3\pi x_3}{b}\right)\sin\left(\frac{4\pi x_1}{b}\right)\\
&-\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{2\pi x_3}{b}\right)\sin\left(\frac{3\pi x_1}{b}\right)\sin\left(\frac{4\pi x_4}{b}\right)-\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_1}{b}\right)\sin\left(\frac{3\pi x_4}{b}\right)\sin\left(\frac{4\pi x_2}{b}\right)\Big]
\end{aligned}
$$

1st excited state, $E = \dfrac{39\hbar^2\pi^2}{2mb^2}$

$$
\begin{aligned}
\psi = \frac{1}{b^2\sqrt{6}}\Big[&\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{2\pi x_2}{b}\right)\sin\left(\frac{3\pi x_3}{b}\right)\sin\left(\frac{5\pi x_4}{b}\right)+\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{2\pi x_3}{b}\right)\sin\left(\frac{3\pi x_4}{b}\right)\sin\left(\frac{5\pi x_1}{b}\right)\\
&+\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{3\pi x_1}{b}\right)\sin\left(\frac{5\pi x_2}{b}\right)+\sin\left(\frac{\pi x_4}{b}\right)\sin\left(\frac{2\pi x_1}{b}\right)\sin\left(\frac{3\pi x_2}{b}\right)\sin\left(\frac{5\pi x_3}{b}\right)\\
&-\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{2\pi x_1}{b}\right)\sin\left(\frac{3\pi x_3}{b}\right)\sin\left(\frac{5\pi x_4}{b}\right)-\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{2\pi x_3}{b}\right)\sin\left(\frac{3\pi x_4}{b}\right)\sin\left(\frac{5\pi x_2}{b}\right)\\
&-\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{3\pi x_2}{b}\right)\sin\left(\frac{5\pi x_1}{b}\right)-\sin\left(\frac{\pi x_4}{b}\right)\sin\left(\frac{2\pi x_2}{b}\right)\sin\left(\frac{3\pi x_1}{b}\right)\sin\left(\frac{5\pi x_3}{b}\right)\\
&+\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{2\pi x_3}{b}\right)\sin\left(\frac{3\pi x_2}{b}\right)\sin\left(\frac{5\pi x_4}{b}\right)+\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_2}{b}\right)\sin\left(\frac{3\pi x_4}{b}\right)\sin\left(\frac{5\pi x_1}{b}\right)\\
&+\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{3\pi x_1}{b}\right)\sin\left(\frac{5\pi x_3}{b}\right)+\sin\left(\frac{\pi x_4}{b}\right)\sin\left(\frac{2\pi x_1}{b}\right)\sin\left(\frac{3\pi x_3}{b}\right)\sin\left(\frac{5\pi x_2}{b}\right)\\
&-\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{2\pi x_3}{b}\right)\sin\left(\frac{3\pi x_4}{b}\right)\sin\left(\frac{5\pi x_2}{b}\right)-\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{3\pi x_2}{b}\right)\sin\left(\frac{5\pi x_1}{b}\right)\\
&-\sin\left(\frac{\pi x_4}{b}\right)\sin\left(\frac{2\pi x_2}{b}\right)\sin\left(\frac{3\pi x_1}{b}\right)\sin\left(\frac{5\pi x_3}{b}\right)-\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{2\pi x_1}{b}\right)\sin\left(\frac{3\pi x_3}{b}\right)\sin\left(\frac{5\pi x_4}{b}\right)\\
&+\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{3\pi x_3}{b}\right)\sin\left(\frac{5\pi x_2}{b}\right)+\sin\left(\frac{\pi x_4}{b}\right)\sin\left(\frac{2\pi x_3}{b}\right)\sin\left(\frac{3\pi x_2}{b}\right)\sin\left(\frac{5\pi x_1}{b}\right)\\
&+\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_2}{b}\right)\sin\left(\frac{3\pi x_1}{b}\right)\sin\left(\frac{5\pi x_4}{b}\right)+\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{2\pi x_1}{b}\right)\sin\left(\frac{3\pi x_4}{b}\right)\sin\left(\frac{5\pi x_3}{b}\right)\\
&-\sin\left(\frac{\pi x_1}{b}\right)\sin\left(\frac{2\pi x_4}{b}\right)\sin\left(\frac{3\pi x_2}{b}\right)\sin\left(\frac{5\pi x_3}{b}\right)-\sin\left(\frac{\pi x_4}{b}\right)\sin\left(\frac{2\pi x_2}{b}\right)\sin\left(\frac{3\pi x_3}{b}\right)\sin\left(\frac{5\pi x_1}{b}\right)\\
&-\sin\left(\frac{\pi x_2}{b}\right)\sin\left(\frac{2\pi x_3}{b}\right)\sin\left(\frac{3\pi x_1}{b}\right)\sin\left(\frac{5\pi x_4}{b}\right)-\sin\left(\frac{\pi x_3}{b}\right)\sin\left(\frac{2\pi x_1}{b}\right)\sin\left(\frac{3\pi x_4}{b}\right)\sin\left(\frac{5\pi x_2}{b}\right)\Big]
\end{aligned}
$$

Solutions provided by:
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