NATIONAL UNIVERSITY OF SINGAPORE

PC3130 Quantum Mechanics II

(Semester II: AY 2010-11)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **4** questions and comprises **6** printed pages, including a Table of the Clebsch-Gordan coefficients.
2. Answer **any 3** questions.
3. All questions carry equal marks.
4. Answers to the questions are to be written in the answer books.
5. This is a CLOSED BOOK examination.

1. (a) A hydrogen atom can be viewed as a proton and an electron with the Coulomb interacting potential between them.

Show that the Schrodinger equation for such as a system can be separated into two parts, namely, one describing the motion of the center of mass and another describing the relative motion of the electron and the proton.

(b) Derive the spherical coordinate representation of the 3rd component ℓ_3 of the orbital angular momentum of a particle, namely, $\ell_3 = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$

Note that $x_1 = r\sin\theta\cos\phi,\quad x_2 = r\sin\theta\sin\phi,\quad x_3 = r\cos\theta$.

Hence show that the following commutation relation holds

$$\left[\phi,\ \frac{\hbar}{i}\frac{\partial}{\partial \phi}\right] = i\hbar\ .$$

Explain briefly whether this commutation relation can be rewritten as

$$[\phi,\ \ell_3] = i\hbar\ ?$$

Justify your explanation.

Making use of the expression

$$\ell_{+} = \hbar\, e^{i\phi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right]$$

show that an explicit expression for the spherical harmonics $Y_\ell^m(\theta,\phi)$ when m takes its maximum value $+\ell$ is given by

$$Y_\ell^\ell(\theta,\phi) = N\ \sin^\ell\theta\ \ e^{i\ell\phi}$$

where N is a normalization constant.

2. (a) Write down the Hamiltonian of the particle in the presence of electric and magnetic fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$. Note that $\underset{\sim}{E} = -\nabla A_0 - \frac{\partial \underset{\sim}{A}}{\partial t}$ and $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$ in the usual notations.

Show that $\frac{d\langle \underset{\sim}{x} \rangle}{dt} = \frac{1}{m}\left\langle \left(\underset{\sim}{p} - q\,\underset{\sim}{A} \right) \right\rangle$.

Note that you can assume without proof

$$\frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar}\langle [H, Q] \rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle .$$

(b) Consider a particle of charge q moving with velocity $\underset{\sim}{v}$ through electric and magnetic fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$. The $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are given by $\underset{\sim}{E} = -2C x_3\, \underset{\sim}{k}$ and $\underset{\sim}{B} = (0, 0, B_0) = B_0\, \underset{\sim}{k}$, where B_0, C are constants and $\underset{\sim}{k}$ unit vector of third axis of the coordinate frame.

(i)Show that the vector potentials associated with the fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are

$$\underset{\sim}{A} = \frac{B_0}{2}(x_1\, \underset{\sim}{j} - x_2\, \underset{\sim}{i}), \quad \text{and} \quad A_0 = C x_3^2,$$

Note that $\underset{\sim}{E} = -\nabla A_0 - \frac{\partial \underset{\sim}{A}}{\partial t}$ and $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$

(ii)Show that the allowed energy, for a particle of mass m and charge q, in these fields is given by

$$E(n_1, n_2) = (n_1 + \frac{1}{2})\hbar\omega_1 + (n_2 + \frac{1}{2})\hbar\omega_2, \quad (n_1, n_2 = 0,1,2,\ldots),$$

where $\omega_1 \equiv qB_0/m$ and $\omega_2 \equiv \sqrt{2qC/m}$.

Hence or otherwise deduce the Landau levels of the particle, the quantum analog to cyclotron motion.

3. (a) Show that for an infinitesimal rotation about an unit vector $\underset{\sim}{u}$ by an angle ε, the rotation operator $\Re_{\underset{\sim}{u}}(\varepsilon)$ in the three dimensional physical space can be written as

$$\Re_{\underset{\sim}{u}}(\varepsilon) = 1 + \varepsilon \underset{\sim}{u} \wedge .$$

Let $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ be the three unit vectors forming a right-handed Cartesian system. Show that the infinitesimal rotation

$$\Re \equiv \Re_{\underset{\sim}{v}}^{-1}(\varepsilon)\Re_{\underset{\sim}{u}}^{-1}(\varepsilon)\Re_{\underset{\sim}{v}}(\varepsilon)\Re_{\underset{\sim}{u}}(\varepsilon)$$

differs from $\Re_{\underset{\sim}{w}}(-\varepsilon^2)$ only by terms of higher order than ε^2.

The following formula can be assumed without proof

$$\Re_{\underset{\sim}{u}}(\varepsilon) = 1 + \varepsilon \underset{\sim}{u} \wedge + \frac{1}{2!}\varepsilon^2 \underset{\sim}{u} \wedge \underset{\sim}{u} \wedge$$

(b) Obtain a complete set of simultaneous normalized eigenvectors of $\underset{\sim 1}{J}^2$, $\underset{\sim 2}{J}^2$, $\underset{\sim}{J}^2$ and J_3, where $\underset{\sim}{J} = \underset{\sim 1}{J} + \underset{\sim 2}{J}$, given simultaneous normalized eigenvectors $|j_1 m_1>$ and $|j_2 m_2>$ of $\underset{\sim 1}{J}^2$, J_{13} and $\underset{\sim 2}{J}^2$, J_{23} respectively such that

$$\underset{\sim 1}{J}^2 |j_1 m_1> = \frac{3}{4}\hbar^2 |j_1 m_1>,$$

$$\underset{\sim 2}{J}^2 |j_2 m_2> = 6\hbar^2 |j_2 m_2>.$$

You may make use of the Table of Clebsch-Gordan Coefficients on page 6.

4. (a) Consider two angular momentum operators $\underset{\sim 1}{J}$ and $\underset{\sim 2}{J}$. The total angular momentum operator is defined by $\underset{\sim}{J} = \underset{\sim 1}{J} + \underset{\sim 2}{J}$. Show that the allowed quantum number j associated with the square of the total angular momentum $\underset{\sim}{J}^2$ is given

by

$$j = j_1 + j_2,\ j_1 + j_2 - 1,\ \ldots,\ |j_1 - j_2|$$

where j_i is the quantum number associated with the square of the angular momentum $\underset{\sim}{J}_i$, $i = 1, 2$.

(b) Consider a system of four non-interacting particles, each of mass m, that are confined in a one-dimensional infinite potential well

$$\begin{aligned} V(x) &= 0 \qquad && 0 < x < b \\ &= \infty && \text{elsewhere} \end{aligned}$$

The one-particle states are $\psi = \sqrt{\frac{2}{b}}\ \sin\left(\frac{n\pi x}{b}\right)$, with energies $E_n = \frac{\hbar^2 \pi^2}{2mb^2} n^2$,

where $n = 1, 2, 3, \ldots$.

Find the eigenfunctions and the corresponding energies of the ground state and the first excited states of the four-particle system.

You should distinguish the three cases: (i) distinguishable identical particles, (ii) identical bosons(ignoring spins) and (iii) identical fermions(ignoring spins).

- The End -

CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A $\sqrt{\ }$ is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

m_1 m_2	J J ... / M M ...
m_1 m_2	Coefficients
⋮ ⋮	

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	1, 0	0, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1, −1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2, +3/2
+1	+1/2	1

m_1	m_2	3/2, +1/2	1/2, +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2, −1/2	1/2, −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2, −3/2
−1	−1/2	1

2×1/2

m_1	m_2	5/2, +5/2
+2	1/2	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2, +1/2	3/2, +1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2, −1/2	3/2, −1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2, −3/2	3/2, −3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2, −5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2, +2
+3/2	+1/2	1

m_1	m_2	2, +1	1, +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2, 0	1, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2, −1	1, −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2, −2
−3/2	−1/2	1

2×1

m_1	m_2	3, +3
+2	+1	1

m_1	m_2	3, +2	2, +2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3, +1	2, +1	1, +1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	6/15	−1/2	1/10

m_1	m_2	3, 0	2, 0	1, 0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3, −1	2, −1	1, −1
0	−1	6/15	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3, −2	2, −2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3, −3
−2	−1	1

1×1

m_1	m_2	2, +2
+1	+1	1

m_1	m_2	2, +1	1, +1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2, 0	1, 0	0, 0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2, −1	1, −1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2, −2
−1	−1	1

3/2×1

m_1	m_2	5/2, +5/2
+3/2	+1	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2, −3/2	3/2, −3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2, −5/2
−3/2	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

3/2×3/2

m_1	m_2	3, +3
+3/2	+3/2	1

m_1	m_2	3, +2	2, +2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	3, +1	2, +1	1, +1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	3, 0	2, 0	1, 0	0, 0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	3, −1	2, −1	1, −1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	3, −2	2, −2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	3, −3
−3/2	−3/2	1

$$d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2} \qquad d^1_{0,0} = \cos\theta$$

2×3/2

m_1	m_2	7/2, +7/2
+2	+3/2	1

m_1	m_2	7/2, +5/2	5/2, +5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	7/2, +3/2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/7	16/35	2/5
+1	1/2	4/7	1/35	−2/5
0	3/2	2/7	−18/35	1/5

m_1	m_2	7/2, +1/2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	1/2	18/35	−3/35	−1/5	1/5
−1	3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	7/2, −1/2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	1/2	12/35	−5/14	0	3/10
−2	3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	7/2, −3/2	5/2, −3/2	3/2, −3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	1/2	1/7	−16/35	2/5

m_1	m_2	7/2, −5/2	5/2, −5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	7/2, −7/2
−2	−3/2	1

2×2

m_1	m_2	4, +4
+2	+2	1

m_1	m_2	4, +3	3, +3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	4, +2	3, +2	2, +2
+2	0	3/14	1/2	2/7
+1	1	4/7	0	−3/7
0	2	3/14	−1/2	2/7

m_1	m_2	4, +1	3, +1	2, +1	1, +1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	1	3/7	−1/5	−1/14	3/10
−1	2	1/14	−3/10	3/7	−1/5

m_1	m_2	4, 0	3, 0	2, 0	1, 0	0, 0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	1	8/35	−2/5	1/14	1/10	−1/5
−2	2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	4, −1	3, −1	2, −1	1, −1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	1	1/14	−3/10	3/7	−1/5

m_1	m_2	4, −2	3, −2	2, −2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	4, −3	3, −3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	4, −4
−2	−2	1

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta+1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 20.1: Sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The signs and numbers in the current tables have been calculated by computer programs written independently by Cohen and at LBL.