PC3130 (AY2012/2013 sem 2)
Suggested solutions

Throughout this solution sheet, the notation will follow Einstein summation convention unless otherwise specified.

Question 1

1a. (AY2012/13 sem 2 notes, Chapter 1.1.2, page 14) Orbital angular momentum is defined as $\vec{L} = \vec{x} \times \vec{p}$, i.e. it has components $L_i = \varepsilon_{ijk} x_j p_k$. It has the commutation relation $[L_i, L_j] = i\hbar\varepsilon_{ijk} L_k$.

1ai.
$$\begin{aligned}
\left[L_i, L^2\right] = [L_i, L_j L_j] &= [L_i, L_j] L_j + L_j [L_i, L_j] \\
&= (i\hbar\varepsilon_{ijk} L_k) L_j + L_j (i\hbar\varepsilon_{ijk} L_k) \\
&= i\hbar (\varepsilon_{ijk} L_k L_j + \varepsilon_{ij'k'} L_{j'} L_{k'}) \text{ (relabelling indices, } j \to j',\ k \to k') \\
&= i\hbar (\varepsilon_{ijk} L_k L_j - \varepsilon_{ik'j'} L_{j'} L_{k'}) \\
&= i\hbar (\varepsilon_{ijk} L_k L_j - \varepsilon_{ijk} L_k L_j) \text{ (relabelling indices, } k' \to j,\ j' \to k) \\
&= 0
\end{aligned}$$

1aii.
$$\begin{aligned}
[L_i, x_j] = [\varepsilon_{ilm} x_l p_m, x_j] &= \varepsilon_{ilm} (x_l [p_m, x_j] + [x_l, x_j] p_m) \\
&= \varepsilon_{ilm} (x_l (-i\hbar\delta_{mj}) + 0) \text{ since } x_l \text{ and } x_j \text{ commute} \\
&= -i\hbar\varepsilon_{ilj} x_l \\
&= i\hbar\varepsilon_{ijl} x_l \\
&= i\hbar\varepsilon_{ijk} x_k \text{ (relabelling index, } l \to k)
\end{aligned}$$

1aiii.
$$\begin{aligned}
[L_i, p_j] = [\varepsilon_{ilm} x_l p_m, p_j] &= \varepsilon_{ilm} (x_l [p_m, p_j] + [x_l, p_j] p_m) \\
&= \varepsilon_{ilm} (0 + (i\hbar\delta_{lj}) p_m) \text{ since } p_m \text{ and } p_j \text{ commute} \\
&= i\hbar\varepsilon_{ijm} p_m \\
&= i\hbar\varepsilon_{ijk} p_k \text{ (relabelling index, } m \to k)
\end{aligned}$$

$$\begin{aligned}
\left[L_3, x^2\right] = [L_3, x_j x_j] &= [L_3, x_j] x_j + x_j [L_3, x_j] \\
&= (i\hbar\varepsilon_{3jk} x_k) x_j + x_j (i\hbar\varepsilon_{3jk} x_k) \\
&= i\hbar (\varepsilon_{3jk} x_k x_j + \varepsilon_{3j'k'} x_{j'} x_{k'}) \text{ (relabelling indices, } j \to j',\ k \to k') \\
&= i\hbar (\varepsilon_{3jk} x_k x_j - \varepsilon_{3k'j'} x_{j'} x_{k'}) \\
&= i\hbar (\varepsilon_{3jk} x_k x_j - \varepsilon_{3jk} x_k x_j) \text{ (relabelling indices, } k' \to j,\ j' \to k) \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\left[L_3, p^2\right] = [L_3, p_j p_j] &= [L_3, p_j]\, p_j + p_j\, [L_3, p_j] \\
&= (i\hbar\varepsilon_{3jk} p_k)\, p_j + p_j\, (i\hbar\varepsilon_{3jk} p_k) \\
&= i\hbar\left(\varepsilon_{3jk} p_k p_j + \varepsilon_{3j'k'} p_{j'} p_{k'}\right) \ \text{(relabelling indices, } j \to j',\ k \to k'\text{)} \\
&= i\hbar\left(\varepsilon_{3jk} p_k p_j - \varepsilon_{3k'j'} p_{j'} p_{k'}\right) \\
&= i\hbar\left(\varepsilon_{3jk} p_k p_j - \varepsilon_{3jk} p_k p_j\right) \ \text{(relabelling indices, } k' \to j,\ j' \to k\text{)} \\
&= 0
\end{aligned}
$$

Since L_3 commutes with x^2 and p^2 as shown above, it also commutes with any analytic function of x^2 or p^2. Hence we have $\left[L_3, \dfrac{p^2}{2m}\right] = 0$ and $[L_3, V(r)] = \left[L_3, V\left(\sqrt{x^2}\right)\right] = 0$, and therefore $[L_3, H] = \left[L_3, \frac{p^2}{2m} + V(r)\right] = [L_3, V(r)] + \left[L_3, \frac{p^2}{2m}\right] = 0$.

1b. Since $[L_i, L^2] = 0$ (as shown in part 1ai), we also have $[L_i^2, L^2] = 0$. Also, by definition $L^2 = L_1^2 + L_2^2 + L_3^2$. Therefore,

$$
\begin{aligned}
\left[L_1^2, L_2^2\right] = \left[L_1^2, L^2 - L_1^2 - L_3^2\right] &= \left[L_1^2, L^2\right] - \left[L_1^2, L_1^2\right] - \left[L_1^2, L_3^2\right] \\
&= 0 - 0 - \left[L_1^2, L_3^2\right] \\
&= \left[L_3^2, L_1^2\right]
\end{aligned}
$$

By symmetry (permuting indices) we also get $[L_3^2, L_1^2] = [L_2^2, L_3^2]$, thus overall we have $[L_1^2, L_2^2] = [L_2^2, L_3^2] = [L_3^2, L_1^2]$.

Within the $l = 1$ subspace, the $|1, m\rangle$ states with $m = -1, 0, 1$ form a complete basis. We shall therefore show that the above commutators vanish for all angular momentum states with $l = 1$ by showing $\langle 1, m'| [L_1^2, L_3^2] |1, m\rangle = 0$ for all $m, m' = -1, 0, 1$ (this necessarily implies that the commutators vanish in the $l = 1$ subspace because any state in this subspace can be expressed as some linear combination of the $|1, m\rangle$ states with $m = -1, 0, 1$).

We first recall that in terms of the raising/lowering operators, we have $L_1 = \dfrac{L_+ + L_-}{2}$ and $L_2 = \dfrac{L_+ - L_-}{2i}$. Therefore we have

$$
\begin{aligned}
L_1^2 &= \left(\frac{L_+ + L_-}{2}\right)^2 = \frac{1}{4}\left(L_+^2 + L_-^2 + L_+L_- + L_-L_+\right) \\
L_2^2 &= \left(\frac{L_+ - L_-}{2i}\right)^2 = -\frac{1}{4}\left(L_+^2 + L_-^2 - L_+L_- - L_-L_+\right)
\end{aligned}
$$

To reduce clutter in the following steps, we also note that within the $l = 1$ subspace, the "normalisation constants" $\hbar\sqrt{l(l+1) - m(m \pm 1)}$ for the raising/lowering operators $L_\pm$ take the value $\hbar\sqrt{2}$ in all non-zero cases (as seen by substituting $l = 1$, $m = -1, 0, 1$ into the expression), i.e. we have $L_\pm |1, m\rangle = \hbar\sqrt{2}\, |1, m \pm 1\rangle$ in all non-zero cases.

For all m, m' within the $l = 1$ subspace,

$$\begin{aligned}\langle 1, m'| \left[L_1^2, L_3^2\right] |1, m\rangle = \langle 1, m'| L_1^2 L_3^2 - L_3^2 L_1^2 |1, m\rangle &= \langle 1, m'| L_1^2 L_3^2 |1, m\rangle - \langle 1, m'| L_3^2 L_1^2 |1, m\rangle \\ &= \langle 1, m'| L_1^2 m^2 |1, m\rangle - \langle 1, m'| m'^2 L_1^2 |1, m\rangle \\ &= \left(m^2 - m'^2\right) \langle 1, m'| L_1^2 |1, m\rangle\end{aligned}$$

We now consider two cases, $m = \pm m'$ and $m \neq \pm m'$.

Case 1 ($m = \pm m'$): We have $(m^2 - m'^2) = 0$ and therefore $\langle 1, m'| [L_1^2, L_3^2] |1, m\rangle = 0$.

Case 2 ($m \neq \pm m'$): We first note that this necessarily implies exactly one of m, m' is 0 (and the other is ± 1), since the only values they can take are $-1, 0, 1$. If $m = 0$, then $L_\pm^2 |1, m\rangle = 0$ (since applying the raising/lowering operator twice to the $m = 0$ state in the $l = 1$ subspace gives zero) and thus $\langle 1, m'| L_\pm^2 |1, m\rangle = 0$. If $m' = 0$, then since $L_\pm^2 |1, \mp 1\rangle = 2\hbar^2 |1, \pm 1\rangle$ and $L_\pm^2 |1, \pm 1\rangle = 0$, we must have $\langle 1, m'| L_\pm^2 |1, m\rangle = 0$ (because the $|1, \pm 1\rangle$ states are orthogonal to the $|1, 0\rangle$ state).

(Alternative for $m' = 0$: L_+ and L_- are the Hermitian conjugates of each other (since L_1 and L_2 are Hermitian), and thus $\langle 1, 0| L_\pm^2 = \left(\left(L_\pm^2\right)^\dagger |1, 0\rangle\right)^\dagger = \left(L_\mp^2 |1, 0\rangle\right)^\dagger = 0$. Therefore $\langle 1, m'| L_\pm^2 |1, m\rangle = 0$ for $m' = 0$.)

Thus for this case,

$$\begin{aligned}\langle 1, m'| L_1^2 |1, m\rangle &= \frac{1}{4} \langle 1, m'| \left(L_+^2 + L_-^2 + L_+ L_- + L_- L_+\right) |1, m\rangle \\ &= \frac{1}{4} \left(0 + 0 + \langle 1, m'| L_+ L_- |1, m\rangle + \langle 1, m'| L_- L_+ |1, m\rangle\right) \\ &= \frac{1}{4} \left((2\hbar^2 \text{ or } 0)\langle 1, m'|1, m\rangle + (0 \text{ or } 2\hbar^2)\langle 1, m'|1, m\rangle\right) \\ &= 0 \quad \text{since } m \neq \pm m' \text{ implies } m \neq m' \text{ and thus } \langle 1, m'|1, m\rangle = 0\end{aligned}$$

and therefore $\langle 1, m'| [L_1^2, L_3^2] |1, m\rangle = 0$.

Since the cases considered cover all possibilities, we have $\langle 1, m'| [L_1^2, L_3^2] |1, m\rangle = 0$ for all m, m' within the $l = 1$ subspace. Therefore $[L_1^2, L_3^2]$ vanishes within this subspace, and so do the other two commutators (since they are equal).

The $|1, -1\rangle$, $|1, 0\rangle$ and $|1, 1\rangle$ states form a complete basis for the $l = 1$ subspace, and we shall hence express the common eigenstates of L_1^2, L_2^2 and L_3^2 (common eigenstates exist because these operators commute, as shown above) in terms of these states. As there are three states in the basis, the subspace has dimension 3.

We first note that $|1, -1\rangle$, $|1, 0\rangle$ and $|1, 1\rangle$ are eigenstates of L_3^2 with eigenvalues $\hbar^2$, 0 and $\hbar^2$ respectively, i.e. $|1, -1\rangle$ and $|1, 1\rangle$ have the same eigenvalue, while $|1, 0\rangle$ has a different one. Since they form a basis, one of the common eigenstates must be simply $|1, 0\rangle$ itself, because any linear combination of $|1, 0\rangle$ with $|1, -1\rangle$ and/or $|1, 1\rangle$ would *not* be an eigenstate of L_3^2 (unless the coefficients for $|1, -1\rangle$ and $|1, 1\rangle$, or $|1, 0\rangle$, are zero); this is easily seen by applying L_3^2 to such a linear combination (noting the different eigenvalues).

By a similar line of reasoning, the remaining two common eigenstates must be linear combinations of $|1,-1\rangle$ and $|1,1\rangle$ only, because adding a non-zero $|1,0\rangle$ component to such a linear combination would render it not an eigenstate of L_3^2. We also note that since $|1,-1\rangle$ and $|1,1\rangle$ have the same eigenvalue (with respect to L_3^2), any linear combination of them is automatically an eigenstate of L_3^2. It thus only remains to find out what linear combinations of the form $a\,|1,1\rangle + b\,|1,-1\rangle$ are eigenstates of both L_1^2 and L_2^2. (a and b may be complex, but one of them can be chosen to be real and positive without loss of generality, because the global phase has no physical significance.)

To be an eigenstate of L_1^2, we require

$$
\begin{aligned}
&L_1^2\left(a\,|1,1\rangle + b\,|1,-1\rangle\right) = \lambda\left(a\,|1,1\rangle + b\,|1,-1\rangle\right)\\
&\frac{1}{4}\left(L_+^2 + L_-^2 + L_+L_- + L_-L_+\right)\left(a\,|1,1\rangle + b\,|1,-1\rangle\right) = \lambda a\,|1,1\rangle + \lambda b\,|1,-1\rangle\\
&\frac{1}{4}\left(0 + 2\hbar^2 b\,|1,1\rangle + 2\hbar^2 a\,|1,-1\rangle + 0 + 2\hbar^2 a\,|1,1\rangle + 0 + 0 + 2\hbar^2 b\,|1,-1\rangle\right) = \lambda a\,|1,1\rangle + \lambda b\,|1,-1\rangle\\
&\frac{\hbar^2}{2}\left((a+b)\,|1,1\rangle + (a+b)\,|1,-1\rangle\right) = \lambda a\,|1,1\rangle + \lambda b\,|1,-1\rangle
\end{aligned}
$$

This gives us the system of equations $\frac{\hbar^2}{2}(a+b) = \lambda a$, $\frac{\hbar^2}{2}(a+b) = \lambda b$. Subtracting the second equation from the first, we get $\lambda(a-b) = 0$, with the solutions

$a = b$: Then $\frac{\hbar^2}{2}(a+a) = \lambda a$, and so $\lambda = \hbar^2$. Choosing a to be real and positive without loss of generality, and normalising the state, we have $a = \frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$.

$\lambda = 0$: Then $\frac{\hbar^2}{2}(a+b) = 0$, and so $a = -b$. Choosing a to be real and positive without loss of generality, and normalising the state, we have $a = \frac{1}{\sqrt{2}}$, $b = -\frac{1}{\sqrt{2}}$.

Therefore, the linear combinations of $|1,-1\rangle$ and $|1,1\rangle$ that are also eigenstates of L_1^2 are $\frac{1}{\sqrt{2}}\,|1,-1\rangle \pm \frac{1}{\sqrt{2}}\,|1,1\rangle$ (or scalar multiples thereof). They have different eigenvalues with respect to L_1^2, are orthogonal to each other and $|1,0\rangle$ (and thus together with $|1,0\rangle$ must be a basis for this subspace, as it has dimension 3), and we have shown earlier that $|1,0\rangle$ must be one of the common eigenstates of L_1^2, L_2^2 and L_3^2; hence, they must be precisely the common eigenstates we are looking for (the two common eigenstates other than $|1,0\rangle$ cannot be a linear combination of both these two states, because such a combination would not be an eigenstate of L_1^2 due to the different eigenvalues). (Alternatively, one can explicitly verify this by applying L_2^2 to these states and showing they are indeed eigenstates of L_2^2.)

In summary, the common eigenstates of L_1^2, L_2^2 and L_3^2 are $|1,0\rangle$, $\frac{1}{\sqrt{2}}\,|1,-1\rangle + \frac{1}{\sqrt{2}}\,|1,1\rangle$ and $\frac{1}{\sqrt{2}}\,|1,-1\rangle - \frac{1}{\sqrt{2}}\,|1,1\rangle$ (or scalar multiples thereof).

(Remark: It can be verified that with respect to each of the operators L_1^2, L_2^2 and L_3^2, two of the common eigenstates have eigenvalue $\hbar^2$ and one has eigenvalue 0, as expected by symmetry. However, the state with eigenvalue 0 is different for each operator.)

Question 2

2ai. The magnetic field in this case is constant and uniform, thus as stated in the question, a possible magnetic vector potential is simply

$$\vec{A} = \frac{1}{2}\vec{B} \times \vec{x} = \frac{1}{2}(0, 0, B_0) \times (x_1, x_2, x_3) = \frac{B_0}{2}(-x_2, x_1, 0)$$

(Alternatively, by definition any vector field $\vec{A}$ that satisfies $\nabla \times \vec{A} = \vec{B}$ is a valid vector potential, thus we can simply verify explicitly that $\nabla \times \vec{A} = \nabla \times \frac{B_0}{2}(-x_2, x_1, 0) = (0, 0, B_0)$.)

We note also that this choice of magnetic vector potential follows the Coulomb gauge (i.e. $\nabla \cdot \vec{A} = 0$), since $\nabla \cdot \frac{B_0}{2}(-x_2, x_1, 0) = \frac{B_0}{2}\left(\partial_1(-x_2) + \partial_2(x_1) + \partial_3(0)\right) = 0$.

2aii. This derivation is essentially identical to that presented in lecture (AY2012/13 sem 2 notes, Chapter 2) apart from the presence of an electric field in the z-direction, so some steps have been skimmed over for brevity. Refer to notes for full details. Note, however, that the energy levels as given in this question do not appear to be entirely correct unless additional constraints are specified on the particle's x_3-motion, as described later.

The Hamiltonian is $\frac{(\vec{p}-q\vec{A})^2}{2m} + V(\vec{x})$. We note that since $\vec{p} = -i\hbar\nabla$, we have by product rule $\vec{p} \cdot \left(\vec{A}\psi\right) = \left(\vec{p} \cdot \vec{A}\right)\psi + \vec{A} \cdot (\vec{p}\psi)$. The $\left(\vec{p} - q\vec{A}\right)^2$ term thus expands to give

$$\begin{aligned}
\left(\vec{p} - q\vec{A}\right)^2 &= \vec{p}^2 + q^2\vec{A}^2 - q\vec{p}\cdot\vec{A} - q\vec{A}\cdot\vec{p} \\
&= \vec{p}^2 + q^2\vec{A}^2 - q\left(\vec{p}\cdot\vec{A}\right) - 2q\vec{A}\cdot\vec{p} \\
&= \vec{p}^2 + q^2\vec{A}^2 - 0 - 2q\vec{A}\cdot\vec{p} \quad \text{since } \left(\vec{p}\cdot\vec{A}\right) = -i\hbar\nabla\cdot\vec{A} = 0 \text{ (Coulomb gauge)} \\
&= \vec{p}^2 + q^2\vec{A}^2 - 2q\left(\frac{1}{2}\vec{B}\times\vec{x}\right)\cdot\vec{p} \\
&= \vec{p}^2 + q^2\vec{A}^2 - q\vec{B}\cdot(\vec{x}\times\vec{p}) \\
&= \vec{p}^2 + q^2\vec{A}^2 - q\vec{B}\cdot\vec{L} \\
&= \vec{p}^2 + q^2\frac{B_0^2}{4}\left((-x_2)^2 + x_1^2 + 0^2\right) - q\,(0, 0, B_0)\cdot(L_1, L_2, L_3) \\
&= \vec{p}^2 + \frac{q^2B_0^2}{4}\left(x_1^2 + x_2^2\right) - qB_0L_3
\end{aligned}$$

Since in this case $V(\vec{x}) = 0$, the Hamiltonian can thus be expressed as

$$\begin{aligned}
H &= \frac{1}{2m}\left(\vec{p}^2 + \frac{q^2B_0^2}{4}\left(x_1^2 + x_2^2\right) - qB_0L_3\right) + 0 \\
&= H_{2D} + H_{free} - \frac{qB_0}{2m}L_3 \\
&= \left[H_{2D} - \frac{qB_0}{2m}\hbar\left(N_+ - N_-\right)\right] + H_{free}
\end{aligned}$$

where $H_{2D} = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{q^2 B_0^2}{8m}\left(x_1^2 + x_2^2\right)$ is the Hamiltonian of a 2-dimensional harmonic oscillator (in the x_1-x_2 plane) with angular frequency $\omega = \sqrt{\frac{q^2 B_0^2}{4m^2}} = \frac{qB_0}{2m}$, and $H_{free} = \frac{p_3^2}{2m}$ is the Hamiltonian of a 1-dimensional free particle (along the x_3 axis).

Since the Hamiltonian can be separated into a sum of a Hamiltonian for motion in the x_1-x_2 plane (this includes the $\frac{qB_0}{2m}L_3$ term) and a Hamiltonian for motion along the x_3 axis, the motions are uncoupled. We thus use the $|n_+, n_-, k_3\rangle$ basis, where k_3 is the wavenumber associated with the free-particle motion along the x_3 axis. Using this basis, the eigenvalues of H_{2D} are $(n_+ + n_- + 1)\,\hbar\omega$, while the eigenvalues of $-\frac{qB_0}{2m}\hbar\,(N_+ - N_-)$ are $-\frac{qB_0}{2m}\hbar\,(n_+ - n_-)$.

However, the eigenvalues of H_{free} can take any non-negative real value (namely, $\frac{\hbar^2 k_3^2}{2m}$). To eliminate this energy, we need to either confine the particle to a plane parallel to the x_1-x_2 plane, or change to a reference frame such that the x_3-component of the particle energy vanishes. With this constraint, the eigenvalues of the Hamiltonian (and hence the allowed energies) are then

$$
\begin{aligned}
E = (n_+ + n_- + 1)\,\hbar\omega - \frac{qB_0}{2m}\hbar\,(n_+ - n_-) &= (n_+ + n_- + 1)\,\hbar\frac{qB_0}{2m} - (n_+ - n_-)\,\hbar\frac{qB_0}{2m} \\
&= (2n_- + 1)\,\hbar\frac{qB_0}{2m} \\
&= \left(n_- + \frac{1}{2}\right)\hbar\frac{qB_0}{m}
\end{aligned}
$$

In summary, the allowed energies are thus $E = \left(n + \frac{1}{2}\right)\hbar\omega_1$ where $\omega_1 = \frac{qB_0}{m}$ and n is a non-negative integer, as desired.

2b. Consider the electric dipole moment under the coordinate transformation $\vec{x} \to -\vec{x}$. Since this is only a change of coordinates, it does not affect the physical quantity $\langle nlm|\,\vec{d}\,|nlm\rangle$. (An alternative approach is to note that space inversion is a unitary operation, and thus we have $\langle nlm|'\,\vec{d}'\,|nlm\rangle' = \langle nlm|\,U^\dagger U \vec{d} U^\dagger U\,|nlm\rangle = \langle nlm|\,\vec{d}\,|nlm\rangle$.) Therefore, we have

$$
\begin{aligned}
\langle nlm|\,\vec{d}\,|nlm\rangle = \langle nlm|'\,\vec{d}'\,|nlm\rangle' &= \langle nlm|'\,q\vec{x}'\,|nlm\rangle' \\
&= q\,\langle nlm|\,(-1)^l\,(-\vec{x})\,(-1)^l\,|nlm\rangle \\
&= q(-1)^{2l+1}\,\langle nlm|\,\vec{x}\,|nlm\rangle \\
&= -q\,\langle nlm|\,\vec{x}\,|nlm\rangle \\
&= -\,\langle nlm|\,\vec{d}\,|nlm\rangle
\end{aligned}
$$

Since $\langle nlm|\,\vec{d}\,|nlm\rangle = -\,\langle nlm|\,\vec{d}\,|nlm\rangle$, we must thus have $\langle nlm|\,\vec{d}\,|nlm\rangle = 0$.

The third component of the electric dipole moment of the $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|2,0,0\rangle + |2,1,0\rangle\right)$ state is given by

$$\begin{aligned}
\langle\psi|\, d_3\, |\psi\rangle &= \frac{1}{2}\left(\langle 2,0,0| + \langle 2,1,0|\right) d_3 \left(|2,0,0\rangle + |2,1,0\rangle\right) \\
&= \frac{1}{2}\left(\langle 2,0,0|\, d_3\, |2,0,0\rangle + \langle 2,0,0|\, d_3\, |2,1,0\rangle + \langle 2,1,0|\, d_3\, |2,0,0\rangle + \langle 2,1,0|\, d_3\, |2,1,0\rangle\right) \\
&= \frac{1}{2}\left(\langle 2,0,0|\, d_3\, |2,1,0\rangle + \langle 2,1,0|\, d_3\, |2,0,0\rangle\right) \text{ since } \langle nlm|\, \vec{d}\, |nlm\rangle = 0 \text{ as shown earlier} \\
&= \frac{1}{2}\left(\langle 2,0,0|\, d_3\, |2,1,0\rangle + \langle 2,0,0|\, d_3\, |2,1,0\rangle^*\right) \text{ since } x_3 \text{ is Hermitian (an observable)}
\end{aligned}$$

(Alternatively, the two terms could be evaluated individually if desired)

$$= \operatorname{Re}\left(\langle 2,0,0|\, d_3\, |2,1,0\rangle\right)$$

We thus just need to evaluate $\langle 2,0,0|\, d_3\, |2,1,0\rangle$, i.e. $\langle 2,0,0|\, qx_3\, |2,1,0\rangle$:

$$\begin{aligned}
\langle 2,0,0|\, qx_3\, |2,1,0\rangle &= \langle 2,0,0|\, qr\cos\theta\, |2,1,0\rangle \\
&= q\int \left(R_{2,0}(r)Y_0^0(\theta,\phi)\right)^* r\cos\theta \left(R_{2,1}(r)Y_1^0(\theta,\phi)\right)\, \mathrm{d}^3r \\
&= q\int_{4\pi}\int_0^\infty \left(R_{2,0}(r)Y_0^0(\theta,\phi)\right)^* r\cos\theta \left(R_{2,1}(r)Y_1^0(\theta,\phi)\right) r^2\, \mathrm{d}r\mathrm{d}\Omega \\
&= q\left(\int_{4\pi} Y_0^0(\theta,\phi)^* \cos\theta\, Y_1^0(\theta,\phi)\, \mathrm{d}\Omega\right)\left(\int_0^\infty R_{2,0}(r)^* R_{2,1}(r) r^3\, \mathrm{d}r\right) \\
&= q\,\langle 0,0|\cos\theta\,|1,0\rangle \left(\int_0^\infty R_{2,0}(r)R_{2,1}(r)r^3\, \mathrm{d}r\right) \quad \text{since } R_{2,0}(r) \text{ is real} \\
&= q\sqrt{\frac{(1+0)(1-0)}{(2+1)(2-1)}}\left(-\frac{9}{\sqrt{3}}a_0\right) \quad \text{by the given formulae} \\
&= q\sqrt{\frac{1}{3}}\left(-\frac{9}{\sqrt{3}}a_0\right) \\
&= -3qa_0
\end{aligned}$$

Therefore $\langle\psi|\, d_3\, |\psi\rangle = \operatorname{Re}\left(\langle 2,0,0|\, d_3\, |2,1,0\rangle\right) = -3qa_0$, as desired.

Important note: If you wish to evaluate $\int_0^\infty R_{2,0}(r)R_{2,1}(r)r^3\,\mathrm{d}r$ explicitly rather than use the provided result, note that the normalisation factor given in the question for the $R_{2,1}(r)$ function is incorrect — it should be $\frac{1}{\sqrt{24}}$, not $\frac{1}{\sqrt{2}}$.

Question 3

3a. This derivation is essentially identical to that presented in lecture (AY2012/13 sem 2 notes, Chapter 4), so some steps have been skimmed over for brevity. Refer to notes for full details.

Consider an arbitrary vector $\vec{A}$ rotated about the axis $\hat{n}$ by an infinitesimal angle ε. Let the change be denoted $\delta\vec{A}$, and the angle between $\vec{A}$ and $\hat{n}$ be denoted θ.

The direction of $\delta\vec{A}$ is perpendicular to both $\vec{A}$ and $\hat{n}$, and thus can be specified by the unit vector $\dfrac{\hat{n}\times\vec{A}}{|\hat{n}\times\vec{A}|} = \dfrac{\hat{n}\times\vec{A}}{|\vec{A}|\sin\theta}$.

The magnitude of $\delta\vec{A}$ can be seen by geometry to be $\left(|\vec{A}|\sin\theta\right)\varepsilon$.

Therefore we have $\delta\vec{A} = \dfrac{\hat{n}\times\vec{A}}{|\vec{A}|\sin\theta}\left(|\vec{A}|\sin\theta\right)\varepsilon = \varepsilon\hat{n}\times\vec{A}$, in other words $\vec{A}' = \vec{A} + \varepsilon\hat{n}\times\vec{A} = (1+\varepsilon\hat{n}\times)\,\vec{A}$. The rotation operator $\Re_{\hat{n}}(\varepsilon)$ in physical space is thus $\Re_{\hat{n}}(\varepsilon) = 1 + \varepsilon\hat{n}\times$.

To find the rotation operator in Hilbert space, we use $\psi'(\vec{x}') = \psi(\vec{x})$ and apply a Taylor expansion since ε is infinitesimal:

$$\psi'(\vec{x}') = \psi(\vec{x}) = \psi(\Re_{\hat{n}}(\varepsilon)^{-1}\vec{x}') = \psi(\vec{x}' - \varepsilon\hat{n}\times\vec{x}') = \psi(\vec{x}') - \varepsilon(\hat{n}\times\vec{x}')\cdot\nabla\psi(\vec{x}') + O(\varepsilon^2)$$

Relabelling $\vec{x}'$ as $\vec{x}$, and recalling $\vec{p} = \frac{\hbar}{i}\nabla$ and $\vec{L} = \vec{x}\times\vec{p}$, we obtain

$$\begin{aligned}\psi'(\vec{x}) = \psi(\vec{x}) - \varepsilon(\hat{n}\times\vec{x})\cdot\nabla\psi(\vec{x}) + O(\varepsilon^2) &= \psi(\vec{x}) - \frac{i}{\hbar}\varepsilon\left((\hat{n}\times\vec{x})\cdot\vec{p}\right)\psi(\vec{x}) + O(\varepsilon^2) \\ &= \psi(\vec{x}) - \frac{i}{\hbar}\varepsilon\left(\hat{n}\cdot\vec{L}\right)\psi(\vec{x}) + O(\varepsilon^2)\end{aligned}$$

Since $\psi'(\vec{x}) = \langle\vec{x}|\,R\,|\psi\rangle$, the coordinate representation R_c of R is hence $R_c = 1 - \dfrac{i}{\hbar}\varepsilon\left(\hat{n}\cdot\vec{L}\right)$.

We now consider a multicomponent wavefunction $\psi_i(\vec{x})$. We then have $\psi_i'(\vec{x}') = \pi_{ij}\psi_j(\vec{x})$. Repeating the previous procedure gives

$$\psi_i'(\vec{x}) = \pi_{ij}\left[\psi_j(\vec{x}) - \frac{i}{\hbar}\varepsilon\left(\hat{n}\cdot\vec{L}\right)\psi_j(\vec{x}) + O(\varepsilon^2)\right]$$

We assume that for infinitesimal rotations, $\pi_{ij} = \delta_{ij} - \dfrac{i}{\hbar}\varepsilon\left(\hat{n}\cdot\vec{S}\right)_{ij}$. This hence gives

$$\begin{aligned}\psi_i'(\vec{x}) &= \left[\delta_{ij} - \frac{i}{\hbar}\varepsilon\left(\hat{n}\cdot\vec{S}\right)_{ij}\right]\left[\psi_j(\vec{x}) - \frac{i}{\hbar}\varepsilon\left(\hat{n}\cdot\vec{L}\right)\psi_j(\vec{x}) + O(\varepsilon^2)\right] \\ &= \psi_i(\vec{x}) - \frac{i}{\hbar}\varepsilon\left(\hat{n}\cdot\vec{L}\right)\psi_i(\vec{x}) - \frac{i}{\hbar}\varepsilon\left(\hat{n}\cdot\vec{S}\right)_{ij}\psi_j(\vec{x}) + O(\varepsilon^2) \\ &= \left(1 - \frac{i}{\hbar}\varepsilon\left(\hat{n}\cdot\vec{L}\right) - \frac{i}{\hbar}\varepsilon\left(\hat{n}\cdot\vec{S}\right)\right)_{ij}\psi_j(\vec{x}) + O(\varepsilon^2)\end{aligned}$$

Defining $\vec{J} = \vec{L} + \vec{S}$ and keeping terms to order ε, we have $\psi'_i(\vec{x}) = \left(1 - \frac{i}{\hbar}\varepsilon\left(\hat{n}\cdot\vec{J}\right)\right)_{ij}\psi_j(\vec{x})$, and therefore $R_{\hat{n}}(\varepsilon) = 1 - \frac{i}{\hbar}\varepsilon\left(\hat{n}\cdot\vec{J}\right)$.

To find the finite rotation operator, we use the fact that $R_{\hat{n}}(\theta+\delta\theta) = R_{\hat{n}}(\delta\theta)R_{\hat{n}}(\theta)$. Letting $\delta\theta$ be infinitesimal and applying the previous result, we hence have

$$R_{\hat{n}}(\theta+\delta\theta) = \left(1 - \frac{i}{\hbar}\delta\theta\left(\hat{n}\cdot\vec{J}\right)\right)R_{\hat{n}}(\theta) = R_{\hat{n}}(\theta) - \frac{i}{\hbar}\delta\theta\left(\hat{n}\cdot\vec{J}\right)R_{\hat{n}}(\theta)$$

Rearranging terms gives us $\dfrac{R_{\hat{n}}(\theta+\delta\theta) - R_{\hat{n}}(\theta)}{\delta\theta} = -\dfrac{i}{\hbar}\left(\hat{n}\cdot\vec{J}\right)R_{\hat{n}}(\theta)$

$$\implies \frac{d}{d\theta}R_{\hat{n}}(\theta) = -\frac{i}{\hbar}\left(\hat{n}\cdot\vec{J}\right)R_{\hat{n}}(\theta)$$

$$\implies R_{\hat{n}}(\theta) = \exp\left(-\frac{i}{\hbar}\theta\left(\hat{n}\cdot\vec{J}\right)\right)$$

3b. Since the expectation value of an observable should remain unchanged under rotation, we have $\langle\psi|\,Q\,|\psi\rangle = \langle\psi'|\,Q'\,|\psi'\rangle$, i.e. $\langle\psi|\,Q\,|\psi\rangle = \langle\psi|\,R^\dagger Q' R\,|\psi\rangle$. Since this holds for any arbitrary state $|\psi\rangle$, this implies $Q = R^\dagger Q' R$. As rotation is unitary, we have $R^{-1} = R^\dagger$, and thus $Q = R^\dagger Q' R \implies Q' = RQR^\dagger$.

By geometric considerations, we would expect J_2 to transform to $J_2\cos\theta - J_1\sin\theta$ under a rotation of angle θ about the x_3-axis. We shall now proceed to verify this algebraically, using power series expansions. Noting that $R_{\hat{k}}(-\theta) = e^{\frac{i}{\hbar}\theta J_3} = R_{\hat{k}}(\theta)^\dagger$, we shall ease the intermediate calculations slightly by instead showing that $R_{\hat{k}}(\theta)^\dagger J_2 R_{\hat{k}}(\theta) = J_2\cos\theta + J_1\sin\theta$, then substituting $\theta \to -\theta$ to obtain the desired result $R_{\hat{k}}(\theta)J_2R_{\hat{k}}(\theta)^\dagger = J_2\cos\theta - J_1\sin\theta$.

We first note that $\frac{d}{d\theta}R_{\hat{k}}(\theta) = \frac{d}{d\theta}e^{-\frac{i}{\hbar}\theta J_3} = -\frac{i}{\hbar}J_3R_{\hat{k}}(\theta)$, and similarly $\frac{d}{d\theta}R_{\hat{k}}(\theta)^\dagger = \frac{i}{\hbar}J_3R_{\hat{k}}(\theta)^\dagger$.

Also, since $R_{\hat{k}}(\theta)$ has a power series expansion in terms of J_3 and no other operators, it commutes with J_3 (similarly for $R_{\hat{k}}(\theta)^\dagger$).

Denote $R_{\hat{k}}(\theta)^\dagger J_2 R_{\hat{k}}(\theta)$ as $f(\theta)$. We shall show by induction that

$$f^{(n)}(\theta) = \begin{cases} (-1)^{\frac{n}{2}}R_{\hat{k}}(\theta)^\dagger J_2 R_{\hat{k}}(\theta) & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}}R_{\hat{k}}(\theta)^\dagger J_1 R_{\hat{k}}(\theta) & \text{if } n \text{ is odd} \end{cases}$$

Denoting the above statement as $P(n)$, we shall show the base case $P(0)$ to be true, then show that for any non-negative even number m, we have $P(m) \implies P(m+1) \implies P(m+2)$, thereby proving $P(n)$ to be true for all $n \in \mathbb{Z}_{\geq 0}$ by induction.

Base case $P(0)$: $f(\theta) = (-1)^0 R_{\hat{k}}(\theta)^\dagger J_2 R_{\hat{k}}(\theta) = R_{\hat{k}}(\theta)^\dagger J_2 R_{\hat{k}}(\theta)$ is indeed true.

Inductive step: Assume $P(m)$ is true for some non-negative even m. Then we have $f^{(m)}(\theta) = (-1)^{\frac{m}{2}} R_{\hat{k}}(\theta)^\dagger J_2 R_{\hat{k}}(\theta)$, and thus

$$
\begin{aligned}
f^{(m+1)}(\theta) &= \frac{d}{d\theta}\left((-1)^{\frac{m}{2}} R_{\hat{k}}(\theta)^\dagger J_2 R_{\hat{k}}(\theta)\right) \\
&= (-1)^{\frac{m}{2}} \left(\left(\frac{d}{d\theta} R_{\hat{k}}(\theta)^\dagger\right) J_2 R_{\hat{k}}(\theta) + R_{\hat{k}}(\theta)^\dagger J_2 \left(\frac{d}{d\theta} R_{\hat{k}}(\theta)\right)\right) \\
&= (-1)^{\frac{m}{2}} \left(\left(\frac{i}{\hbar} J_3 R_{\hat{k}}(\theta)^\dagger\right) J_2 R_{\hat{k}}(\theta) + R_{\hat{k}}(\theta)^\dagger J_2 \left(-\frac{i}{\hbar} J_3 R_{\hat{k}}(\theta)\right)\right) \\
&= (-1)^{\frac{m}{2}} \left(\frac{i}{\hbar}\right) \left(R_{\hat{k}}(\theta)^\dagger J_3 J_2 R_{\hat{k}}(\theta) - R_{\hat{k}}(\theta)^\dagger J_2 J_3 R_{\hat{k}}(\theta)\right) \text{ since } R_{\hat{k}}(\theta)^\dagger \text{ commutes with } J_3 \\
&= (-1)^{\frac{m}{2}} \left(\frac{i}{\hbar}\right) R_{\hat{k}}(\theta)^\dagger \left(J_3 J_2 - J_2 J_3\right) R_{\hat{k}}(\theta) \\
&= (-1)^{\frac{m}{2}} \left(\frac{i}{\hbar}\right) R_{\hat{k}}(\theta)^\dagger \left(-i\hbar J_1\right) R_{\hat{k}}(\theta) \text{ since } [J_3, J_2] = -i\hbar J_1 \\
&= (-1)^{\frac{m}{2}} R_{\hat{k}}(\theta)^\dagger J_1 R_{\hat{k}}(\theta)
\end{aligned}
$$

therefore $P(m+1)$ is true (noting that $m+1$ is odd). We also hence have, similarly,

$$
\begin{aligned}
f^{(m+2)}(\theta) &= \frac{d}{d\theta}\left((-1)^{\frac{m}{2}} R_{\hat{k}}(\theta)^\dagger J_1 R_{\hat{k}}(\theta)\right) \\
&= (-1)^{\frac{m}{2}} \left(\left(\frac{i}{\hbar} J_3 R_{\hat{k}}(\theta)^\dagger\right) J_1 R_{\hat{k}}(\theta) + R_{\hat{k}}(\theta)^\dagger J_1 \left(-\frac{i}{\hbar} J_3 R_{\hat{k}}(\theta)\right)\right) \\
&= (-1)^{\frac{m}{2}} \left(\frac{i}{\hbar}\right) R_{\hat{k}}(\theta)^\dagger \left(J_3 J_1 - J_1 J_3\right) R_{\hat{k}}(\theta) \text{ since } R_{\hat{k}}(\theta)^\dagger \text{ commutes with } J_3 \\
&= (-1)^{\frac{m+2}{2}} R_{\hat{k}}(\theta)^\dagger J_2 R_{\hat{k}}(\theta) \text{ since } [J_3, J_1] = i\hbar J_2
\end{aligned}
$$

therefore $P(m+2)$ is true (noting that $m+2$ is even). Hence we have shown that for any non-negative even m, we have $P(m) \implies P(m+1) \implies P(m+2)$.

Thus by induction, $P(n)$ is true for all $n \in \mathbb{Z}_{\geq 0}$. We note that this in turn gives us

$$
\begin{aligned}
f^{(n)}(0) &= \begin{cases} (-1)^{\frac{n}{2}} R_{\hat{k}}(0)^\dagger J_2 R_{\hat{k}}(0) & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} R_{\hat{k}}(0)^\dagger J_1 R_{\hat{k}}(0) & \text{if } n \text{ is odd} \end{cases} \\
&= \begin{cases} (-1)^{\frac{n}{2}} J_2 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} J_1 & \text{if } n \text{ is odd} \end{cases} \quad \text{since } R_{\hat{k}}(0)^\dagger = R_{\hat{k}}(0) = 1
\end{aligned}
$$

Finally, we consider the power series expansion of $f(\theta)$ (since the exponential function is analytic) to obtain

$$
\begin{aligned}
f(\theta) = \sum_{n=0}^{\infty}\left(\frac{\theta^n}{n!}f^{(n)}(0)\right) &= \sum_{\text{even } n}^{\infty}\left(\frac{\theta^n}{n!}(-1)^{\frac{n}{2}}J_2\right) + \sum_{\text{odd } n}^{\infty}\left(\frac{\theta^n}{n!}(-1)^{\frac{n-1}{2}}J_1\right) \\
&= J_2\sum_{\text{even } n}^{\infty}\left(\frac{\theta^n}{n!}(-1)^{\frac{n}{2}}\right) + J_1\sum_{\text{odd } n}^{\infty}\left(\frac{\theta^n}{n!}(-1)^{\frac{n-1}{2}}\right) \\
&= J_2\cos\theta + J_1\sin\theta
\end{aligned}
$$

Therefore we have shown that $R_{\hat{k}}(\theta)^\dagger J_2 R_{\hat{k}}(\theta) = J_2\cos\theta + J_1\sin\theta$. Substituting $\theta \to -\theta$ gives us $R_{\hat{k}}(\theta) J_2 R_{\hat{k}}(\theta)^\dagger = J_2\cos\theta - J_1\sin\theta$, as expected.

(Remark: If known, the identity $e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]]\ldots$ (see the Baker-Campbell-Hausdorff formula, which is related) can be used instead to derive this desired result. The identity itself may be demonstrated by various methods such as differentiating with respect to a parameter, or expanding e^A and e^{-A} as power series and multiplying terms appropriately, or by an argument similar to the above procedure.)

Using the power series expansion of the exponential function, and recalling that $R_{\hat{k}}(\theta)$ is unitary (i.e. $R_{\hat{k}}(\theta)^\dagger R_{\hat{k}}(\theta) = R_{\hat{k}}(\theta)R_{\hat{k}}(\theta)^\dagger = 1$), we have

$$
\begin{aligned}
R_{\hat{k}}(\pi)e^{-\frac{i}{\hbar}\alpha J_2}R_{\hat{k}}(\pi)^\dagger &= R_{\hat{k}}(\pi)\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{\hbar}\alpha J_2\right)^n\right)R_{\hat{k}}(\pi)^\dagger \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-R_{\hat{k}}(\pi)\frac{i}{\hbar}\alpha J_2 R_{\hat{k}}(\pi)^\dagger\right)^n \quad \text{since } R_{\hat{k}}(\pi)^\dagger R_{\hat{k}}(\pi) = 1 \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{\hbar}\alpha\left(J_2\cos\pi - J_1\sin\pi\right)\right)^n \quad \text{by the previous result} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}\alpha J_2\right)^n \\
&= e^{\frac{i}{\hbar}\alpha J_2}\ \text{, as desired.}
\end{aligned}
$$

(This method generalises to other functions with power series expansions as well.)

Question 4

4a. This derivation is essentially identical to that presented in lecture (AY2012/13 sem 2 notes, Chapter 4), so some steps have been skimmed over for brevity. Refer to notes for full details. To avoid confusion with the particle subscripts, this answer uses subscripts x, y, z instead of 1, 2, 3 to denote the Cartesian components of the angular momenta.

We show that the components of $\vec{J}$ satisfy the angular momentum commutation relations $[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$, hence the eigenvalues of J^2 and J_z can be written as $j(j+1)\hbar^2$ and $m\hbar$ respectively, where j is a non-negative integer or half-integer and m increases in integer steps from $-j$ to j (inclusive).

J_1^2, J_2^2, J_{1z} and J_{2z} commute and thus can have common eigenstates $|\alpha, j_1, j_2, m_1, m_2\rangle$. We show that $J_z|\alpha, j_1, j_2, m_1, m_2\rangle = (m_1+m_2)\hbar|\alpha, j_1, j_2, m_1, m_2\rangle$ (where $J_z = J_{1z} + J_{2z}$) and thus $|\alpha, j_1, j_2, m_1, m_2\rangle$ are also eigenstates of J_z, with eigenvalue $(m_1+m_2)\hbar$. Therefore, we have $m = m_1 + m_2$.

We now use a counting method to find the allowed values of j in terms of j_1 and j_2. Let c_m be the number of states $|j, m\rangle$ for a fixed m, and d_j be the number of states $|j, m\rangle$ for a fixed m and j. We note that $d_j = 1$ if and only if that particular combination of j, m is allowed, and $d_j = 0$ otherwise. Since $|m| \leq j$ and taking $m \geq 0$, we have

$c_m = d_m + d_{m+1} + d_{m+2} + ...$
$c_{m+1} = d_{m+1} + d_{m+2} + ...$

and thus $d_m = c_m - c_{m+1}$. Tabulating values of c_m, we have

$\boldsymbol{m_1}$	$\boldsymbol{m_2}$	$\boldsymbol{m = m_1 + m_2}$	$\boldsymbol{c_m}$
j_1	j_2	$j_1 + j_2$	1
j_1 $j_1 - 1$	$j_2 - 1$ j_2	$j_1 + j_2 - 1$	2
j_1 $\vdots$ $j_1 - n$	$j_2 - n$ $\vdots$ j_2	$j_1 + j_2 - n$	$n + 1$

From the table, we have $c_{j_1+j_2} = 1$, $c_{j_1+j_2-1} = 2$ and so on, increasing by one each time, up until a certain point (to be discussed below). Therefore we have $d_{j_1+j_2} = 1 - 0 = 1$, $d_{j_1+j_2-1} = 2 - 1 = 1$, and so on, up until that point, i.e. those values of j are allowed.

c_m stops increasing when the possible values of m_1 in the first column and/or m_2 in the second column cover all values they can take (i.e. all values in integer steps from $-j_1$ to j_1 or $-j_2$ to j_2 respectively), whichever occurs first. It can be seen that this occurs after $2j_1$ and/or $2j_2$ steps, whichever is smaller. The value of j at which this occurs is $(j_1+j_2) - \min(2j_1, 2j_2) = |j_1 - j_2|$, thus we have $d_j = 0$ from this point onwards. (Alternative: Draw a diagram as in the example in the lecture notes.)

Therefore, the allowed values of j are $j_1 + j_2, j_1 + j_2 - 1, ..., |j_1 - j_2|$.

4b. We recall that $\langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m\rangle = 0$ whenever $m \neq m_1 + m_2$. For the $\left|1, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}\right\rangle$ state, we have $j_1 = 1$, $j_2 = \frac{1}{2}$ and $m = \frac{3}{2}$. Therefore, the only values of m_1 and m_2 which can satisfy $m_1 + m_2 = m = \frac{3}{2}$ are $m_1 = 1$ and $m_2 = \frac{1}{2}$, since $-j_1 \leq m_1 \leq j_1$ and $-j_2 \leq m_2 \leq j_2$. Hence, the coupled state in terms of the uncoupled states must simply be $\left|1, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}\right\rangle = |1, 1\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle$, since all the other coefficients are zero. (If desired, one can explicitly verify using the formula provided that $\langle 1, \frac{1}{2}, 1, \frac{1}{2} | 1, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}\rangle = \langle 1, \frac{1}{2}, \frac{3}{2} - \frac{1}{2}, \frac{1}{2} | 1, \frac{1}{2}, 1 + \frac{1}{2}, \frac{3}{2}\rangle = \sqrt{\dfrac{1 + \frac{3}{2} + \frac{1}{2}}{2(1) + 1}} = 1$, as expected.)

For the $\left|1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle$ state, we again have $j_1 = 1$, $j_2 = \frac{1}{2}$, but this time $m = -\frac{1}{2}$ can be formed in two ways from sums of the possible values of m_1 and m_2, namely $m_1 = -1, m_2 = \frac{1}{2}$ and $m_1 = 0, m_2 = -\frac{1}{2}$. We evaluate the corresponding Clebsch-Gordan coefficients with the provided formulae:

$$\langle 1, \frac{1}{2}, -1, \frac{1}{2} | 1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\rangle = \langle 1, \frac{1}{2}, -\frac{1}{2} - \frac{1}{2}, \frac{1}{2} | 1, \frac{1}{2}, 1 - \frac{1}{2}, \frac{3}{2}\rangle = -\sqrt{\frac{1 - \left(-\frac{1}{2}\right) + \frac{1}{2}}{2(1) + 1}} = -\sqrt{\frac{2}{3}}$$

$$\langle 1, \frac{1}{2}, 0, -\frac{1}{2} | 1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\rangle = \langle 1, \frac{1}{2}, -\frac{1}{2} + \frac{1}{2}, \frac{1}{2} | 1, \frac{1}{2}, 1 - \frac{1}{2}, \frac{3}{2}\rangle = \sqrt{\frac{1 + \left(-\frac{1}{2}\right) + \frac{1}{2}}{2(1) + 1}} = \sqrt{\frac{1}{3}}$$

The mod-squared sum of these coefficients is 1, as expected. (Again, $\langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m\rangle = 0$ for the other values of m_1, m_2 since $m \neq m_1 + m_2$ for those values.) Therefore, the coupled state in terms of the uncoupled states is $\left|1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\, |1, 0\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\, |1, -1\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle$.

As seen from the above result, the coefficient for the component with the spin-$\frac{1}{2}$ particle (i.e. the electron) in spin-down is $\sqrt{\frac{1}{3}}$. Therefore, the probability of measuring the electron spin to be down is $\frac{1}{3}$.

(Remark: One can verify against a table of Clebsch-Gordan coefficients that these are indeed the correct values for the coefficients.)

4c. For ease of notation, we shall denote the n^{th}-energy-level one-particle wavefunctions by $\psi_n(x_i) = \sqrt{\frac{2}{b}} \sin\left(\frac{n\pi x_i}{b}\right)$.

Since the particles have half-integer spin, they are fermions. Therefore, the state must be antisymmetric under particle interchange, and the particles must all be in different states. As the particles are assumed to all be in the same spin state, the spatial wavefunction must be antisymmetric and the one-particle wavefunctions must all be different.

Therefore, the wavefunction of the ground state ϕ_0 is

$$\phi_0(x_1, x_2, x_3) = \frac{1}{\sqrt{6}}(\psi_1(x_1)\psi_2(x_2)\psi_3(x_3) + \psi_2(x_1)\psi_3(x_2)\psi_1(x_3) + \psi_3(x_1)\psi_1(x_2)\psi_2(x_3)$$
$$-\psi_1(x_1)\psi_3(x_2)\psi_2(x_3) - \psi_3(x_1)\psi_2(x_2)\psi_1(x_3) - \psi_2(x_1)\psi_1(x_2)\psi_3(x_3))$$

with energy $(1^2 + 2^2 + 3^2)\frac{\hbar^2\pi^2}{2mb^2} = 14\frac{\hbar^2\pi^2}{2mb^2}$.

The wavefunction of the first excited state ϕ_1 is

$$\phi_1(x_1, x_2, x_3) = \frac{1}{\sqrt{6}}(\psi_1(x_1)\psi_2(x_2)\psi_4(x_3) + \psi_2(x_1)\psi_4(x_2)\psi_1(x_3) + \psi_4(x_1)\psi_1(x_2)\psi_2(x_3)$$
$$-\psi_1(x_1)\psi_4(x_2)\psi_2(x_3) - \psi_4(x_1)\psi_2(x_2)\psi_1(x_3) - \psi_2(x_1)\psi_1(x_2)\psi_4(x_3))$$

with energy $(1^2 + 2^2 + 4^2)\frac{\hbar^2\pi^2}{2mb^2} = 21\frac{\hbar^2\pi^2}{2mb^2}$.

Solutions provided by: Ernest Tan