NATIONAL UNIVERSITY OF SINGAPORE

PC3130 Quantum Mechanics II

(Semester II: AY 2012-13)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **4** questions and comprises **6** printed pages.
2. Answer **3** questions.
3. All questions carry equal marks.
4. Answers to the questions are to be written in the answer books.
5. This is a CLOSED BOOK examination.

1(a) Define orbital angular momentum ℓ_i and write down its commutation relation.

Show that **(i)** $[\ell_i, \underset{\sim}{\ell}^2]=0$; **(ii)** $[\ell_i, x_j]=i\hbar\,\varepsilon_{ijk}\,x_k$; and **(iii)** $[\ell_i, p_j]=i\hbar\,\varepsilon_{ijk}\,p_k$

Evaluate the commutators $[\ell_3, \underset{\sim}{x}^2]$ and $[\ell_3, \underset{\sim}{p}^2]$ where

$$\underset{\sim}{x}^2 = x_1^2 + x_2^2 + x_3^2 \text{ and } \underset{\sim}{p}^2 = p_1^2 + p_2^2 + p_3^2 .$$

Hence or otherwise, prove that $[\ell_3, H]=0$, where the Hamiltonian is $H=\dfrac{\underset{\sim}{p}^2}{2m}+V(r)$. Thus $H, \ell_3, \underset{\sim}{\ell}^2$ are mutually compatible observables.

(b) Show that $[\ell_1^2, \ell_2^2]=[\ell_2^2, \ell_3^2]=[\ell_3^2, \ell_1^2]$.

Hence, or otherwise, show that these commutators vanish in angular momentum state $|\ell\ m\rangle$ with $\ell=1$.

Find the three common eigenstates of ℓ_1^2, ℓ_2^2 and ℓ_3^2 for this case.

2(a) Consider a particle of charge q moving with velocity $\underset{\sim}{v}$ in the magnetic field $\underset{\sim}{B}$. The $\underset{\sim}{B}$ is given by $\underset{\sim}{B} = (0,0,B_0)= B_0\,\underset{\sim}{k}$, where B_0 is a constant and $\underset{\sim}{k}$ unit vector of third axis of the coordinate frame.

(i) Show that the vector potentials associated with the above field $\underset{\sim}{B}$ can be written as

$$\underset{\sim}{A} = \frac{B_0}{2}(-x_2\,\underset{\sim}{i}+ x_1\,\underset{\sim}{j})=\frac{B_0}{2}(-x_2,\ x_1,\ 0) .$$

Note that $\underset{\sim}{B}=\nabla\wedge\underset{\sim}{A}$, and for constant $\underset{\sim}{B}$, $\underset{\sim}{A}=\frac{1}{2}(\underset{\sim}{B}\wedge\underset{\sim}{x})$.

(ii) Show that the allowed energy for a particle of mass m and charge q in this field is given by $E(n) = (n + \frac{1}{2})\hbar\omega_1$, $(n = 0,1,2,\ldots)$, where $\omega_1 \equiv qB_0 / m$.

(b) The electric dipole moment operator is defined as $\underset{\sim}{d} = q\underset{\sim}{x}$, where q and $\underset{\sim}{x}$ are respectively the charge and the position of the particle. The atomic stationary state of hydrogen atom, denoted by $|n\,\ell\,m\rangle$, has a parity $(-1)^{\ell}$. More explicitly, under the space inversion, $\underset{\sim}{x} \to -\underset{\sim}{x}$, the stationary state in spherical coordinate representation changes from $\langle r\theta\phi | n\ell m\rangle$ to $(-1)^{\ell}\langle r\theta\phi | n\ell m\rangle$.

Show that the stationary state $|n\,\ell\,m\rangle$ has zero electric dipole moment, namely, $< n\ell m | \underset{\sim}{d} | n\ell m > = 0$.

Consider the first excited state energy of the hydrogen atom for which $n = 2$. Suppose the state is $|\psi\rangle = \frac{1}{\sqrt{2}}(|2\,0\,0\rangle + |2\,1\,0\rangle)$, show that this state has a non-vanishing electric dipole moment given by

$$\langle\psi|d_3|\psi\rangle = \langle\psi|q\,x_3|\psi\rangle = \langle\psi|qr\cos\theta|\psi\rangle = -3q\,a_0$$

where d_3 is the 3rd component of the dipole moment $\underset{\sim}{d}$ and a_0 is the Bohr radius.

The following results can be used without proof:

The stationary state $\langle r\theta\phi | n\ell m\rangle = R_{n\ell}(r)\; Y_{\ell}^{m}(\theta,\phi)$.

$$R_{20}(r) = \frac{1}{\sqrt{2}} a_0^{-3/2}\,(1 - \frac{1}{2}\frac{r}{a_0})\,\exp(-r/2a_0); \quad R_{21}(r) = \frac{1}{\sqrt{2}} a_0^{-3/2}\,(\frac{r}{a_0})\,\exp(-r/2a_0)$$

$$\int_0^{\infty} R_{21}(r) R_{20}(r)\, r^3 dr = -\frac{9}{\sqrt{3}} a_0; \qquad \Gamma(n) = \int_0^{\infty} e^{-r}\, r^{n-1}\, dr = (n-1)!$$

$$\langle n, \ell+1, m|\cos\theta|\, n, \ell, n\rangle = \sqrt{\frac{(\ell+1-m)(\ell+1+m)}{(2\ell+1)(2\ell+3)}}$$

$$\langle n, \ell-1, m|\cos\theta|\, n, \ell, n\rangle = \sqrt{\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}}\;.$$

3(a) Show that for an infinitesimal rotation about a unit vector $\underset{\sim}{n}$ by an angle ε, the rotation operator $\Re_{\underset{\sim}{n}}(\varepsilon)$ in the three dimensional physical space is given by the expression

$$\Re_{\underset{\sim}{n}}(\varepsilon) = 1 + \varepsilon\, \underset{\sim}{n} \wedge$$

Briefly outline the steps that lead to corresponding infinitesimal rotation operator $R_{\underset{\sim}{n}}(\varepsilon)$ in Hilbert space, which is given by

$$R_{\underset{\sim}{n}}(\varepsilon) = 1 - \frac{i}{\hbar}\varepsilon\, \underset{\sim}{n} \cdot \underset{\sim}{J}$$

where $\underset{\sim}{J} = \underset{\sim}{l} + \underset{\sim}{s}$ is the angular momentum operator in the usual notations.

Hence or otherwise show that the finite rotation operator by an angle θ is given by

$$R_{\underset{\sim}{n}}(\theta) = \exp\left(-\frac{i}{\hbar}\theta\, \underset{\sim}{n} \cdot \underset{\sim}{J} \right)$$

(b) If R is the rotation operator in Hilbert space and Q an observable, show that the transform of Q under rotation, that is Q', is given by

$$Q' = R\, Q\, R^{+}$$

Here R^{+} is the Hermitian conjugate of R.

Find the transform J_2' of the 2nd component J_2 of the angular momentum operator $\underset{\sim}{J}$ under the rotation about the Ox_3 axis (3rd axis, unit vector $\underset{\sim}{k}$), namely under $R_{\underset{\sim}{k}}(\theta) = e^{-i\theta J_3/\hbar}$.

Hence or otherwise, show that $R_{\underset{\sim}{k}}(\pi)\, e^{-i\alpha J_2/\hbar}\, R_{\underset{\sim}{k}}(\pi)^{+} = e^{+i\alpha J_2/\hbar}$

Note: The exponential of an operator Q is given by the power series expansion:

$$e^{Q} \equiv 1 + Q + (1/2)Q^2 + (1/3!)Q^3 + \ldots$$

4(a) Consider the addition of two angular momentum observables $\underset{\sim}{J}_1$ and $\underset{\sim}{J}_2$, and j_i is the quantum number associated with the square of the angular momentum $\underset{\sim}{J}_i^2$, $i=1,2$. In the $(2j_1+1)\cdot(2j_2+1)$ dimensional space spanned by the ket vectors $|\alpha\, j_1\, j_2\, m_1\, m_2\rangle$ (α, j_1, j_2 fixed, m_1, m_2 variables), briefly outline the steps leading to result that the possible values for the quantum number j associated with the square of total angular momentum operator $\underset{\sim}{J}^2$, $\underset{\sim}{J}=\underset{\sim}{J}_1+\underset{\sim}{J}_2$, are given by

$$j = j_1+j_2,\ j_1+j_2-1,\ \ldots,\ |j_1-j_2|\ .$$

(b) The Clebsch-Gordan coefficients $\langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m\rangle$ for $\underset{\sim}{J} = \underset{\sim}{J}_1 + \underset{\sim}{J}_2$ when $j_2 = \frac{1}{2}$ are given by

$$\left\langle j_1, \frac{1}{2}, m-\frac{1}{2}, \frac{1}{2} \middle| j_1, \frac{1}{2}, j_1 \pm \frac{1}{2}, m \right\rangle = \pm\sqrt{\frac{j_1 \pm m + \frac{1}{2}}{2j_1+1}},$$

$$\left\langle j_1, \frac{1}{2}, m+\frac{1}{2}, -\frac{1}{2} \middle| j_1, \frac{1}{2}, j_1 \pm \frac{1}{2}, m \right\rangle = \sqrt{\frac{j_1 \mp m + \frac{1}{2}}{2j_1+1}}.$$

Here $|j_1, j_2, m_1, m_2\rangle = |j_1 m_1\rangle |j_2 m_2\rangle$ is the uncoupled state and $|j_1, j_2, j, m\rangle$ the coupled state.

Express the coupled states $\left|1, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}\right\rangle$ and $\left|1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle$ in terms of the uncoupled states.

If an electron is in the coupled state $\left|1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle$, what is the probability of finding it with spin down?

(c) Consider a system of three non-interacting identical spin ½ particles that are confined in a one-dimensional infinite potential well

$$V(x) = \begin{cases} 0 & 0 < x < b \\ \infty & \text{elsewhere} \end{cases}$$

The one-particle states are $\psi = \sqrt{\frac{2}{b}} \sin\left(\frac{n\pi x}{b}\right)$, with energies $E_n = \frac{\hbar^2 \pi^2}{2mb^2} n^2$, where $n = 1, 2, 3, \ldots$. For simplicity, the three identical spin ½ particles are assumed to be in the same spin state $\left|\frac{1}{2}, \frac{1}{2}\right\rangle$.

Determine the energy and wavefunction of the ground state, and the first excited state of the three-particle system.

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