NATIONAL UNIVERSITY OF SINGAPORE

PC3231 – ELECTRICITY & MAGNETISM II

(Semester I: AY 2015-16)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name.**
2. This examination paper contains FOUR questions and comprises SIX printed pages.
3. Answer **ALL** questions.
4. Each question carries equal marks.
5. Answers to the questions are to be written in the answer books.
6. Please start each question on a new page.
7. This is a CLOSED BOOK examination.
8. Only non-programmable electronic scientific calculators are permitted for this examination.
9. The last three pages contain a list of formulae.

1. (a) Find the charge and current distributions that would give rise to the following potentials

$$V = 0, \quad \mathbf{A} = \begin{cases} \frac{\mu_0 k}{4c}(ct - |x|)^2\,\hat{\mathbf{z}}, & \text{for } |x| < ct \\ 0, & \text{for } |x| > ct \end{cases}$$

where k is a constant, and $c = 1/\sqrt{\mu_0 \epsilon_0}$.

(b) Two equal but opposite point charges (of magnitude q) are separated by a distance $2a$. By integrating Maxwell's stress tensor over a plane equidistant from the two charges, determine the force of one charge on the other.

2. (a) From the relativistic equation, $\mathbf{F} = \mathrm{d}\mathbf{p}/\mathrm{d}t$, show that

$$\mathbf{F} = \frac{m}{\sqrt{1 - u^2/c^2}}\left[\mathbf{a} + \frac{\mathbf{u}(\mathbf{u} \cdot \mathbf{a})}{c^2 - u^2}\right]$$

where m is the rest mass of the particle and $\mathbf{a} = \mathrm{d}\mathbf{u}/\mathrm{d}t$ is the ordinary acceleration.

(b) A plane wave travels in the z direction and is polarized with its $\mathbf{E}$ vector in the x direction. Its average energy flux is 7 mW/m^2 and its frequency is 100 MHz. Find the root mean square (rms) emf induced in a wire loop of radius 10 cm located in the xz plane.

3. A rectangular wave guide of sides $a = 7.21$ cm and $b = 3.40$ cm is used in the transverse magnetic (TM) mode. It is oriented with its axis along the z-axis. TM modes are modes in which the magnetic field is perpendicular to the direction of propagation. Assume that the walls of the waveguide are perfect conductors.

   (a) Solve

   $$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] \tilde{E}_{0z}(x, y) = 0$$

   subject to the following boundary conditions at the inner wall

   $$\boldsymbol{E}^{\parallel} = 0, \quad \boldsymbol{B}^{\perp} = 0$$

   where the symbols have their usual meaning. Hence, derive the dispersion relation (i.e., the relationship between ω and the wavevector k) for this waveguide.

   (b) Show that the group velocity $v_g = \frac{d\omega}{dk}$ is given by

   $$v_g = c\sqrt{1 - (\omega_{mn}/\omega)^2}$$

   where ω_{mn} is the cutoff frequency for the mode.

   (c) Determine whether TM radiation of angular frequency $\omega = 5.0 \times 10^{10}$ rad s^{-1} will propagate in this wave guide.

   (d) Find the attenuation length, i.e., the distance over which the power drops to e^{-1} of its starting value, for a frequency ω that is *half* the lowest cutoff frequency.

4. An unstable particle has a rest mass m_0 and a lifetime t_0 when at rest. It is observed to be in uniform motion with total energy E_0 in the laboratory until it disintegrates into two particles, each of rest mass $m < m_0/2$.

   (a) Derive an expression for the average distance the parent particle travelled in the laboratory before it disintegrates. Express your answer in terms of E_0, t_0, m_0 and c, where c is the speed of light in vacuum.

   If $E_0 = 2.08 \times 10^{-11}$ J, $t_0 = 2 \times 10^{-8}$ s and $m_0 = 1.1 \times 10^{-28}$ kg, what is the distance travelled (in m)?

   (b) Derive an expression for the speed of each secondary particle in the rest frame of the parent particle, in terms of m, m_0 and c.

   If $m_0 = 1.1 \times 10^{-28}$ kg and $m = 3.67 \times 10^{-29}$ kg, what is the speed (in m/s)?

LHS

Formulae

Cylindrical Coordinates

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Vector Derivatives: Cylindrical

$$d\mathbf{l} = ds\,\hat{\mathbf{s}} + s d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s ds d\phi dz$$

$$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial(s v_s)}{\partial s} + \frac{1}{s}\frac{\partial(v_\phi)}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

$$\nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial(s v_\phi)}{\partial s} - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

$$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

Boundary conditions for linear media

$$\left.\begin{array}{ll} \text{(i) } \epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_f, & \text{(iii) } \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0 \\ \text{(ii) } \mathbf{B}_1^\perp - \mathbf{B}_2^\perp = 0, & \text{(iv) } \frac{1}{\mu_1}\mathbf{B}_1^\parallel - \frac{1}{\mu_2}\mathbf{B}_2^\parallel = \mathbf{K}_f\times\hat{\mathbf{n}} \end{array}\right\}$$

Maxwell Stress tensor

$$T_{ij} \equiv \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$

Constants

$\epsilon_0 = 8.85\times10^{-12}\ \mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi\times10^{-7}\ \mathrm{N/A^2}$	(permeability of free space)
$c = 3.00\times10^{8}\ \mathrm{m/s}$	(speed of light)
$e = 1.60\times10^{-19}\ \mathrm{C}$	(charge of the electron)
$m = 9.11\times10^{-31}\mathrm{kg}$	(mass of the electron)

Spherical Coordinates

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Vector Derivatives: Spherical

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\,\hat{\boldsymbol{\phi}};\ d\tau = r^2\sin\theta dr d\theta d\phi$$

$$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta\, v_\phi)}{\partial\theta} - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial(r v_\phi)}{\partial r}\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Vector Identities

Triple Products

1. $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})$
2. $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$

Product Rules

3. $\nabla(fg) = f(\nabla g) + g(\nabla f)$
4. $\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A}$
5. $\nabla\cdot(f\boldsymbol{A}) = f(\nabla\cdot\boldsymbol{A}) + \boldsymbol{A}\cdot(\nabla f)$
6. $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$
7. $\nabla\times(f\boldsymbol{A}) = f(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\times(\nabla f)$
8. $\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla\cdot\boldsymbol{B}) - \boldsymbol{B}(\nabla\cdot\boldsymbol{A})$

Second Derivatives

9. $\nabla\cdot(\nabla\times\boldsymbol{A}) = 0$
10. $\nabla\times(\nabla f) = 0$
11. $\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}$

Retarded and Liénard-Wiechert Potentials

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\boldsymbol{r}',t_r)}{\mathscr{r}}\mathrm{d}\tau',\quad \boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{\mathscr{r}}\mathrm{d}\tau',$$

$$t_r \equiv t - \frac{\mathscr{r}}{c},\quad \mathscr{r} = |\boldsymbol{r}-\boldsymbol{r}'|,$$

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\boldsymbol{v})},\quad \boldsymbol{A}(\boldsymbol{r},t) = \frac{\boldsymbol{v}}{c^2}V(\boldsymbol{r},t),$$

$$|\boldsymbol{\mathscr{r}}| = |\boldsymbol{r} - \boldsymbol{w}(t_r)| = c(t - t_{\text{tr}})$$

Fundamental Theorems

Gradient Theorem: $\int_a^b (\nabla f)\cdot \mathrm{d}\boldsymbol{l} = f(\boldsymbol{b}) - f(\boldsymbol{a})$

Divergence Theorem: $\int (\nabla\cdot\boldsymbol{A})\,\mathrm{d}\tau = \oint \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{a}$

Curl Theorem: $\int (\nabla\times\boldsymbol{A})\cdot\mathrm{d}\boldsymbol{a} = \oint \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}$

Basic Equations of Electrodynamics

In general:

$$\begin{cases} \nabla\cdot\boldsymbol{E} = \frac{1}{\epsilon_0}\rho \\ \nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \\ \nabla\cdot\boldsymbol{B} = 0 \\ \nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla\cdot\boldsymbol{D} = \rho_f \\ \nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \\ \nabla\cdot\boldsymbol{B} = 0 \\ \nabla\times\boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial\boldsymbol{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} \\ \boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M} \end{cases}$$

Linear media:

$$\begin{cases} \boldsymbol{P} = \epsilon_0\chi_e\boldsymbol{E}, & \boldsymbol{D} = \epsilon\boldsymbol{E} \\ \boldsymbol{M} = \chi_m\boldsymbol{H}, & \boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B} \end{cases}$$

Potentials: $\boldsymbol{E} = -\nabla V - \frac{\partial\boldsymbol{A}}{\partial t},\quad \boldsymbol{B} = \nabla\times\boldsymbol{A}$

Lorentz force law: $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$

Lorentz gauge: $\nabla\cdot\boldsymbol{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}.$

Energy: $U = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\mathrm{d}\tau$, Momentm: $\boldsymbol{P} = \epsilon_0\int(\boldsymbol{E}\times\boldsymbol{B})\,\mathrm{d}\tau$

Poynting vector: $\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B})$, Larmor formula: $P = \frac{\mu_0}{6\pi c}q^2a^2$

Vector Analysis

$$\nabla r = \hat{\boldsymbol{r}},\quad \nabla\cdot\left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = 4\pi\delta^3(\boldsymbol{r}),\quad \nabla^2\frac{1}{r} = -4\pi\delta^3(\boldsymbol{r})$$

Monochromatic plane wave

$$\tilde{\boldsymbol{E}}(\boldsymbol{r},t) = \tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\,\hat{\boldsymbol{n}}, \quad \tilde{\boldsymbol{B}}(\boldsymbol{r},t) = \frac{k}{\omega}\tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{n}}) = \frac{k}{\omega}\hat{\boldsymbol{k}}\times\tilde{\boldsymbol{E}}$$

$$\tilde{\boldsymbol{B}}_0 = \frac{k}{\omega}(\hat{\boldsymbol{k}}\times\tilde{\boldsymbol{E}}_0) \text{ in dielectric}, \ \langle u\rangle = \frac{1}{2}\epsilon E_0^2, \ \langle \boldsymbol{g}\rangle = \frac{\langle u\rangle}{c}\hat{\boldsymbol{k}}, \ I = \frac{1}{2}\epsilon v E_{0_I}^2 \cos\theta_I$$

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}, \quad n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} = \sqrt{\epsilon_r\mu_r}, \quad \epsilon = \epsilon_r\epsilon_0, \quad \mu = \mu_r\mu_0$$

$$\tilde{\boldsymbol{B}}_0 = \frac{\tilde{k}}{\omega}(\hat{\boldsymbol{k}}\times\tilde{\boldsymbol{E}}_0) \text{ in conductor}, \quad \tilde{k} = k + i\kappa, \quad d = 1/\kappa \quad \text{skin depth}$$

$$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}, \quad \kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}$$

$$\tilde{k} = Ke^{i\phi}, \quad K = \omega\sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}, \quad \phi \equiv \tan^{-1}(\kappa/k)$$

Hollow rectangular waveguide

$$\omega_{mn} \equiv c\pi\sqrt{(m/a)^2+(n/b)^2} \ \text{cutoff frequency}, \quad k = \frac{1}{c}\sqrt{\omega^2-\omega_{mn}^2}$$

Dipole radiation

$$\text{Electric}: \ \langle P\rangle = \frac{\mu_0 p_0^2\omega^4}{12\pi c}, \quad \text{Magnetic}: \ \langle P\rangle = \frac{\mu_0 m_0^2\omega^4}{12\pi c^3},$$

$$\text{Electric (arbitrary source)}: \ P_{\text{rad}}(t_0) \cong \frac{\mu_0}{6\pi c}[\ddot{p}(t_0)]^2$$

Dirac δ-Function

$$\int_b^c f(t)\,\delta(t-a)\,\mathrm{d}t = f(a), \quad \text{provided } b \le a \le c, \quad \text{otherwise } 0$$

$$\delta(t) = \delta(-t), \quad \delta(at) = \frac{1}{|a|}\delta(t), \quad t\delta(t) = 0$$

Relativity

$$\left.\begin{aligned} \bar{x}^0 &= \gamma(x^0-\beta x^1)\\ \bar{x}^1 &= \gamma(x^1-\beta x^0)\\ \bar{x}^2 &= x^2\\ \bar{x}^3 &= x^3 \end{aligned}\right\} \quad \left.\begin{aligned} \bar{u}_x &= \frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{t}} = \frac{u_x - v}{(1-vu_x/c^2)}\\ \bar{u}_y &= \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{t}} = \frac{u_y}{\gamma(1-vu_x/c^2)}\\ \bar{u}_z &= \frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{t}} = \frac{u_z}{\gamma(1-vu_x/c^2)} \end{aligned}\right\}$$

$$\bar{E}_x = E_x, \ \bar{E}_y = \gamma(E_y - vB_z), \ \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x, \ \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \ \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}, \quad E^2 - p^2c^2 = m^2c^4$$

$$\eta^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \quad p^\mu \equiv m\eta^\mu, \quad E \equiv \frac{mc^2}{\sqrt{1-u^2/c^2}}$$

Miscellaneous

$$\sin(A\pm B) = \sin A\,\cos B \pm \cos A\,\sin B$$

$$\cos(A\pm B) = \cos A\,\cos B \mp \sin A\,\sin B$$

$$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\,\cos\frac{1}{2}(A-B)$$

$$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\,\sin\frac{1}{2}(A-B)$$

$$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\,\cos\frac{1}{2}(A-B)$$

$$\cos A\,\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$$

$$\sin A\,\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

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