NATIONAL UNIVERSITY OF SINGAPORE

PC3231 – ELECTRICITY & MAGNETISM II

(Semester I: AY 2017-18)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name.**
2. This examination paper contains FOUR questions and comprises SIX printed pages.
3. Answer **ALL** questions.
4. Each question carries equal marks.
5. Answers to the questions are to be written in the answer books.
6. Please start each question on a new page.
7. This is a CLOSED BOOK examination.
8. Only non-programmable calculators are permitted for this examination.
9. The last three pages contain a list of formulae.

1. (a) Show that the retarded scalar potential

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}',t_r)}{\mathscr{r}}\mathrm{d}\tau'$$

satisfies the inhomogeneous wave equation

$$\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho$$

where the symbols have their usual meanings.

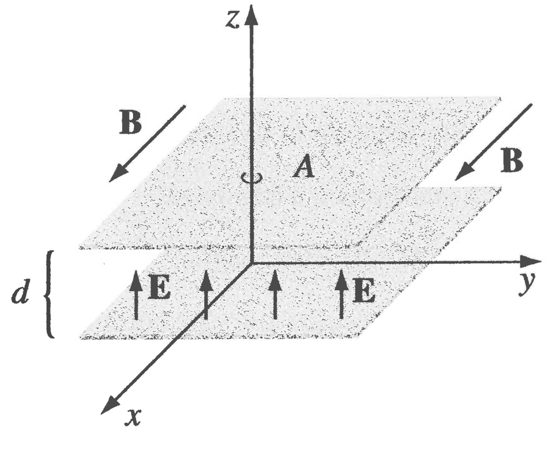
(b) Given the following retarded vector potential,

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{\mathscr{r}}\mathrm{d}\tau'$$

show that the magnetic field is given by

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{\mathscr{r}^2} + \frac{\dot{\boldsymbol{J}}(\boldsymbol{r}',t_r)}{c\mathscr{r}}\right]\times\hat{\boldsymbol{\mathscr{r}}}\,\mathrm{d}\tau'$$

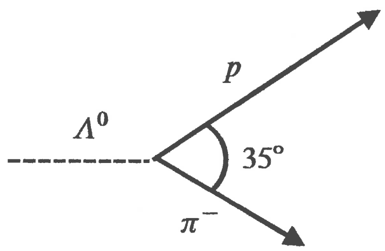
2. A charged parallel-plate capacitor is placed in a uniform magnetic field $\boldsymbol{B} = B\,\hat{\boldsymbol{x}}$. The electric field between the plates is uniform and is pointing along the z direction.



(a) Calculate the electromagnetic momentum in the space between the plates.

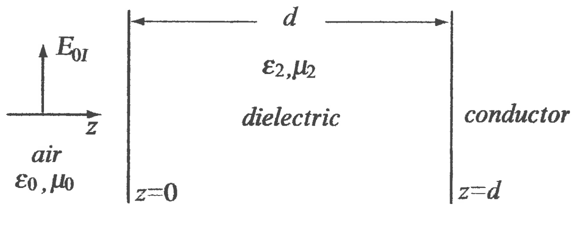
(b) A resistive wire is connected between the plates, along the z axis, so that the capacitor slowly discharges. The current through the wire will experience a magnetic force. Calculate the total impulse delivered to the system during the discharge.

3. (a) A moving baryon decays into a proton and a pion as shown below. Find the rest mass, in MeV, of the baryon, given that the momentum of the proton and the pion are 800-MeV/c and 200-MeV/c, respectively. The rest masses of pion and proton are 140 MeV and 940 MeV.



(b) The star Capella is at a distance of 46 light-years from the Earth. Suppose that an astronaut wants to travel from the Earth to Capella in a time of no more than 20 years as reckoned by clocks aboard his spaceship. At what speed would he have to travel? How long, in years, would the trip take as reckoned by clocks on the Earth? Note that 1 light year is 9.461×10^{15} m.

4. A perfect dielectric of permittivity $\varepsilon_2 = 5\varepsilon_0$ F/m, permeability $\mu_2 = \mu_0$ H/m, and thickness $d = 10$ mm is backed by a perfect conductor (where $\boldsymbol{E} = \boldsymbol{B} = 0$). A plane wave of frequency $f = 900$ MHz is incident from the left as shown:



(a) Assuming that the incident wave is known, calculate the ratio E_{0R}/E_{0I} between the amplitudes of the reflected and incident waves at the interface between free space and dielectric.

(b) Calculate the required (non-zero) thickness d for the reflection to be maximum.

LHS

Formulae

Cylindrical Coordinates

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\boldsymbol{x}} = \cos\phi\,\hat{\boldsymbol{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{y}} = \sin\phi\,\hat{\boldsymbol{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\boldsymbol{s}} = \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases}$$

Vector Derivatives: Cylindrical

$$d\boldsymbol{l} = ds\,\hat{\boldsymbol{s}} + s d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\boldsymbol{z}}; \quad d\tau = s ds d\phi dz$$

$$\nabla t = \frac{\partial t}{\partial s}\hat{\boldsymbol{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\boldsymbol{z}}$$

$$\nabla\cdot\boldsymbol{v} = \frac{1}{s}\frac{\partial(s v_s)}{\partial s} + \frac{1}{s}\frac{\partial(v_\phi)}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

$$\nabla\times\boldsymbol{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\boldsymbol{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial(s v_\phi)}{\partial s} - \frac{\partial v_s}{\partial\phi}\right]\hat{\boldsymbol{z}}$$

$$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

Boundary conditions for linear media

$$\left.\begin{matrix} \text{(i) } \epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_f, & \text{(iii) } \boldsymbol{E}_1^\parallel - \boldsymbol{E}_2^\parallel = 0 \\ \text{(ii) } \boldsymbol{B}_1^\perp - \boldsymbol{B}_2^\perp = 0, & \text{(iv) } \frac{1}{\mu_1}\boldsymbol{B}_1^\parallel - \frac{1}{\mu_2}\boldsymbol{B}_2^\parallel = \boldsymbol{K}_f\times\hat{\boldsymbol{n}} \end{matrix}\right\}$$

Maxwell Stress tensor

$$T_{ij} \equiv \epsilon_0\left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)$$

Constants

$\epsilon_0 = 8.85\times10^{-12}\ \mathrm{C^2/Nm^2}$ (permittivity of free space)
$\mu_0 = 4\pi\times10^{-7}\ \mathrm{N/A^2}$ (permeability of free space)
$c = 3.00\times10^{8}\ \mathrm{m/s}$ (speed of light)
$e = 1.60\times10^{-19}\ \mathrm{C}$ (charge of the electron)
$m = 9.11\times10^{-31}\mathrm{kg}$ (mass of the electron)

Spherical Coordinates

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\boldsymbol{x}} = \sin\theta\cos\phi\,\hat{\boldsymbol{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{y}} = \sin\theta\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\boldsymbol{r}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}} \end{cases}$$

Vector Derivatives: Spherical

$$d\boldsymbol{l} = dr\,\hat{\boldsymbol{r}} + r d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\,\hat{\boldsymbol{\phi}}; \, d\tau = r^2\sin\theta dr d\theta d\phi$$

$$\nabla t = \frac{\partial t}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial t}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\nabla\cdot\boldsymbol{v} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\nabla\times\boldsymbol{v} = \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta\, v_\phi)}{\partial\theta} - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\boldsymbol{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial(r v_\phi)}{\partial r}\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Vector Identities

Triple Products

1. $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$
2. $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$

Product Rules

3. $\nabla(fg) = f(\nabla g) + g(\nabla f)$
4. $\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A}$
5. $\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot (\nabla f)$
6. $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$
7. $\nabla \times (f\boldsymbol{A}) = f(\nabla \times \boldsymbol{A}) - \boldsymbol{A} \times (\nabla f)$
8. $\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A})$

Second Derivatives

9. $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$
10. $\nabla \times (\nabla f) = 0$
11. $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$

Retarded and Liénard-Wiechert Potentials

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}',t_r)}{\mathscr{r}}\mathrm{d}\tau', \quad \boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{\mathscr{r}}\mathrm{d}\tau',$$

$$t_r \equiv t - \frac{\mathscr{r}}{c}, \quad \mathscr{r} = |\boldsymbol{r} - \boldsymbol{r}'|,$$

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \boldsymbol{v})}, \quad \boldsymbol{A}(\boldsymbol{r},t) = \frac{\boldsymbol{v}}{c^2}V(\boldsymbol{r},t),$$

$$|\boldsymbol{\mathscr{r}}| = |\boldsymbol{r} - \boldsymbol{w}(t_r)| = c(t - t_{tr})$$

Fundamental Theorems

Gradient Theorem: $\int_a^b (\nabla f) \cdot \mathrm{d}\boldsymbol{l} = f(\boldsymbol{b}) - f(\boldsymbol{a})$

Divergence Theorem: $\int (\nabla \cdot \boldsymbol{A})\,\mathrm{d}\tau = \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{a}$

Curl Theorem: $\int (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{a} = \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}$

Basic Equations of Electrodynamics

In general:

$$\begin{cases} \nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \boldsymbol{D} = \rho_f \\ \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \\ \boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M} \end{cases}$$

Linear media:

$$\begin{cases} \boldsymbol{P} = \epsilon_0\chi_e \boldsymbol{E}, & \boldsymbol{D} = \epsilon \boldsymbol{E} \\ \boldsymbol{M} = \chi_m \boldsymbol{H}, & \boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B} \end{cases}$$

Potentials: $\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}$

Lorentz force law: $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$

Lorentz gauge: $\nabla \cdot \boldsymbol{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$.

Energy: $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\mathrm{d}\tau$, Momentm: $\boldsymbol{P} = \epsilon_0 \int (\boldsymbol{E} \times \boldsymbol{B})\,\mathrm{d}\tau$

Poynting vector: $\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B})$

Vector Analysis

$$\nabla \mathscr{r} = \hat{\boldsymbol{\mathscr{r}}}, \quad \nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}}), \quad \nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

Monochromatic plane wave

$$\tilde{\boldsymbol{E}}(\boldsymbol{r},t) = \tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\,\hat{\boldsymbol{n}}, \quad \tilde{\boldsymbol{B}}(\boldsymbol{r},t) = \frac{k}{\omega}\tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{n}}) = \frac{k}{\omega}\hat{\boldsymbol{k}}\times\tilde{\boldsymbol{E}}$$

$$\tilde{\boldsymbol{B}}_0 = \frac{k}{\omega}(\hat{\boldsymbol{k}}\times\tilde{\boldsymbol{E}}_0) \text{ in dielectric, } \langle u\rangle = \frac{1}{2}\epsilon E_0^2,\ \langle \boldsymbol{g}\rangle = \frac{\langle u\rangle}{c}\hat{\boldsymbol{k}},\ I = \frac{1}{2}\epsilon v E_{0_I}^2\cos\theta_I$$

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}, \quad n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} = \sqrt{\epsilon_r\mu_r}, \quad \epsilon = \epsilon_r\epsilon_0, \quad \mu = \mu_r\mu_0$$

$$\tilde{\boldsymbol{B}}_0 = \frac{\tilde{k}}{\omega}(\hat{\boldsymbol{k}}\times\tilde{\boldsymbol{E}}_0) \text{ in conductor,} \quad \tilde{k} = k + i\kappa, \quad d = 1/\kappa \quad \text{skin depth}$$

$$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}, \quad \kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}$$

$$\tilde{k} = Ke^{i\phi}, \quad K = \omega\sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}, \quad \phi \equiv \tan^{-1}(\kappa/k)$$

Dipole radiation

$$\text{Electric}: \langle P\rangle = \frac{\mu_0 p_0^2\omega^4}{12\pi c}, \quad \text{Magnetic}: \langle P\rangle = \frac{\mu_0 m_0^2\omega^4}{12\pi c^3},$$

$$\text{Electric (arbitrary source)}: P_{\text{rad}}(t_0) \cong \frac{\mu_0}{6\pi c}[\ddot{p}(t_0)]^2$$

$$\text{Larmor formula: } P = \frac{\mu_0}{6\pi c}q^2a^2$$

Dirac δ-Function

$$\int_b^c f(t)\,\delta(t-a)\,\mathrm{d}t = f(a), \quad \text{provided } b \le a \le c, \quad \text{otherwise } 0$$

$$\delta(t) = \delta(-t), \quad \delta(at) = \frac{1}{|a|}\delta(t), \quad t\delta(t) = 0$$

Relativity

$$\left.\begin{aligned}\bar{x}^0 &= \gamma(x^0 - \beta x^1)\\ \bar{x}^1 &= \gamma(x^1 - \beta x^0)\\ \bar{x}^2 &= x^2\\ \bar{x}^3 &= x^3\end{aligned}\right\} \quad \left\{\begin{aligned}\bar{u}_x &= \frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{t}} = \frac{u_x - v}{(1 - vu_x/c^2)}\\ \bar{u}_y &= \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{t}} = \frac{u_y}{\gamma(1 - vu_x/c^2)}\\ \bar{u}_z &= \frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{t}} = \frac{u_z}{\gamma(1 - vu_x/c^2)}\end{aligned}\right\}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}, \quad E^2 - p^2c^2 = m^2c^4$$

$$\Delta\bar{t} = \Delta t/\gamma \ \text{(time dilation)}, \quad \Delta\bar{x} = \gamma\Delta x \ \text{(length contraction)}$$

$$\bar{E}_x = E_x,\ \bar{E}_y = \gamma(E_y - vB_z),\ \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x,\ \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right),\ \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

$$\bar{\boldsymbol{E}}_\perp = \gamma(\boldsymbol{E}_\perp + \boldsymbol{v}\times\boldsymbol{B}_\perp), \quad \bar{\boldsymbol{B}}_\perp = \gamma\left(\boldsymbol{B}_\perp - \frac{\boldsymbol{v}}{c^2}\times\boldsymbol{E}_\perp\right),$$

$$\bar{\boldsymbol{E}}_\parallel = \boldsymbol{E}_\parallel, \quad \bar{\boldsymbol{B}}_\parallel = \boldsymbol{B}_\parallel, \quad J^\mu \equiv \rho_0\eta^\mu = (\rho c, \boldsymbol{J})$$

$$\eta^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \quad p^\mu \equiv m\eta^\mu = (E/c,\ \boldsymbol{p}), \quad E \equiv \frac{mc^2}{\sqrt{1-u^2/c^2}}$$

Miscellaneous

$$\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

$$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

$$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

$$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

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