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Question 1 i)

Question 1 ii)

Question 1 iii)

Question 2 i)

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0} \qquad (1)$$

$$\left(\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J} \qquad (2)$$

Letting $\vec{A}' = \vec{A} + \vec{\nabla}\lambda, V' = V - \frac{\partial\lambda}{\partial t}$,

(1), $$\nabla^2 V' + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}') = \nabla^2\left(V - \frac{\partial\lambda}{\partial t}\right) + \frac{\partial}{\partial t}[\vec{\nabla}\cdot(\vec{A} + \vec{\nabla}\lambda)]$$

$$= \nabla^2 V - \frac{\partial}{\partial t}\nabla^2\lambda + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) + \frac{\partial}{\partial t}\nabla^2\lambda$$

$$= \nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})$$

Similarly for (2),

$$\left(\nabla^2\vec{A}' - \mu_0\epsilon_0\frac{\partial^2\vec{A}'}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}' + \mu_0\epsilon_0\frac{\partial V'}{\partial t}\right)$$

$$= \left[\nabla^2(\vec{A} + \vec{\nabla}\lambda) - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}(\vec{A} + \vec{\nabla}\lambda)\right] - \vec{\nabla}\left[\vec{\nabla}\cdot(\vec{A} + \vec{\nabla}\lambda) + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(V - \frac{\partial\lambda}{\partial t}\right)\right]$$

$$= \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) + \nabla^2\vec{\nabla}\lambda - \mu_0\epsilon_0\frac{\partial^2\vec{\nabla}\lambda}{\partial t^2} - \vec{\nabla}\nabla^2\lambda + \mu_0\epsilon_0\vec{\nabla}\frac{\partial^2\lambda}{\partial t^2}$$

$$= \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right)$$

$\therefore$ The gauge transformation of Maxwell's equations can be done by adding $\vec{\nabla}\lambda$ to $\vec{A}$ and subtracting $\frac{\partial\lambda}{\partial t}$ from V.

For a point charge,

$$V' = -\frac{q}{4\pi\epsilon_0 r}, \qquad \vec{A}' = 0$$

We let λ to be

$$\lambda = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r}, \qquad \vec{\nabla}\lambda = \frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r}, \qquad \frac{\partial\lambda}{\partial t} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

$$\therefore V = V' - \frac{\partial\lambda}{\partial t} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r} + \frac{1}{4\pi\epsilon_0}\frac{q}{r} = 0$$

$$\vec{A} = \vec{A}' + \vec{\nabla}\lambda = \frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r}, \qquad \text{[shown]}$$

Question 2 ii)

Jefimenko's Equation,

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}(\vec{r}',t_r)}{c\mathscr{r}}\hat{\mathscr{r}} - \frac{\dot{\vec{J}}(\vec{r}',t_r)}{c^2\mathscr{r}}d\tau'$$

When $\vec{J} = \vec{J}(\vec{r}), \dot{\vec{J}} = 0$. The continuity equation,

$$\frac{\partial\rho}{\partial t_r} = -\vec{\nabla}\cdot\vec{J}$$

$$\rho = \left(-\vec{\nabla}\cdot\vec{J}\right)t_r + k$$

$$\rho(\vec{r}',t_r) = \dot{\rho}(\vec{r}',0)t_r + \rho(\vec{r},0) \quad \Rightarrow \quad \dot{\rho}(\vec{r}',t_r) = \dot{\rho}(\vec{r}',0)$$

$$\therefore \vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',0) + \dot{\rho}(\vec{r}',0)t_r}{\mathscr{r}^2} + \frac{\dot{\rho}(\vec{r}',0)}{c\mathscr{r}}d\tau'\hat{\mathscr{r}}$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',0)}{\mathscr{r}^2} + \frac{\dot{\rho}(\vec{r}',0)}{\mathscr{r}^2}\left(t - \frac{\mathscr{r}}{c}\right) + \frac{\dot{\rho}(\vec{r}',0)}{c\mathscr{r}}d\tau'\hat{\mathscr{r}}$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',0)}{\mathscr{r}^2} + \frac{\dot{\rho}(\vec{r}',0)t}{\mathscr{r}^2}d\tau'\hat{\mathscr{r}}$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t)}{\mathscr{r}^2}d\tau'\hat{\mathscr{r}}$$

Question 3 i)

$$E_z = 0, \qquad B_z = B_0\cos\frac{\pi x}{a}$$

$$E_x = 0, \qquad B_x = -\frac{ik}{\left(\frac{\omega}{c}\right)^2 - k^2}\frac{\pi}{a}B_0\sin\frac{\pi x}{a}$$

$$B_y = 0, \qquad E_y = \frac{i\omega}{\left(\frac{\omega}{c}\right)^2 - k^2}\frac{\pi}{a}B_0\sin\frac{\pi x}{a}$$

Question 3 ii)

$$\vec{E} = \begin{pmatrix} 0 \\ E_y e^{i(kz-\omega t)} \\ 0 \end{pmatrix}, \qquad \vec{B} = \begin{pmatrix} B_x e^{i(kz-\omega t)} \\ 0 \\ B_z e^{i(kz-\omega t)} \end{pmatrix}$$

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0}\left(\vec{E} \times \vec{B}^*\right)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \frac{i\omega}{\left(\frac{\omega}{c}\right)^2 - k^2}\frac{\pi}{a}B_0 \sin\frac{\pi x}{a} e^{i(kz-\omega t)} & 0 \\ -\frac{ik}{\left(\frac{\omega}{c}\right)^2 - k^2}\frac{\pi}{a}B_0 \sin\frac{\pi x}{a} e^{i(kz-\omega t)} & 0 & B_0 \cos\frac{\pi x}{a} e^{i(kz-\omega t)} \end{vmatrix}$$

$$= \frac{\omega k \pi^2 B_0^2}{2\mu_0\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]}\frac{1}{a^2}\sin^2\frac{\pi x}{a}\hat{z}$$

Question 3 iii)

$a = 2.28\text{cm}, \qquad b = 1.01\text{cm}, \qquad \omega = 2 \times 10^{10}\text{Hz}$

The group velocity,

$$v_g = c\sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}} = c\sqrt{1 - \frac{c^2\pi^2}{\omega^2}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)}$$

$\therefore v_{g,10} = 2.83 \times 10^8\text{ms}^{-1}, \qquad v_{g,01} = 2.00 \times 10^8\text{ms}^{-1}, \qquad v_{g,11} = 1.75 \times 10^8\text{ms}^{-1}$

Question 4 i)

Skin depth,

$$d = \frac{1}{k_-} = \frac{1}{\omega}\sqrt{\frac{2}{\epsilon\mu}}\frac{1}{\sqrt{\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1}}$$

In a poor conductor, $\sigma \ll \epsilon\omega$,

$$d \approx \frac{1}{\omega}\sqrt{\frac{2}{\epsilon\mu}}\frac{1}{\sqrt{1 - \frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2 - 1}} = \frac{1}{\omega}\sqrt{\frac{2}{\epsilon\mu}}\sqrt{\frac{2\epsilon^2\omega^2}{\sigma^2}} = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$$

Which is independent of frequency.

In a good conductor, $\sigma \gg \epsilon\omega$,

$$d \approx \frac{1}{\omega}\sqrt{\frac{2}{\epsilon\mu}}\frac{1}{\sqrt{\frac{\sigma}{\epsilon\omega}-1}} \approx \frac{1}{\omega}\sqrt{\frac{2}{\epsilon\mu}}\sqrt{\frac{\epsilon\omega}{\sigma}} = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{k_+} = \frac{\lambda}{2\pi}$$

Question 4 ii)

$$\vec{E}_I(z,t) = \tilde{E}_{0I}e^{i(k_1 z-\omega t)}\hat{x}, \qquad \vec{E}_R(z,t) = \tilde{E}_{0R}e^{i(-k_1 z-\omega t)}\hat{x}$$

$$\vec{B}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0I}e^{i(k_1 z-\omega t)}\hat{y}, \qquad \vec{B}_R(z,t) = -\frac{1}{v_1}\tilde{E}_{0R}e^{i(-k_1 z-\omega t)}\hat{y}$$

Using the boundary conditions $\vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0$ and $\frac{1}{\mu_1}\vec{B}_1^{\parallel} - \frac{1}{\mu_2}\vec{B}_2^{\parallel} = 0$,

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \qquad (1)$$

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \tilde{\beta}\tilde{E}_{0T} \qquad (2)$$

where $\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2\omega}\tilde{k}_2$.

Solving both equations, we get

$$\tilde{E}_{0R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0I}$$

Question 4 iii)

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2\omega}\tilde{k}_2 = \frac{c}{\omega}(k_+ + ik_-) \approx \frac{c}{\omega}\sqrt{\frac{\sigma\omega\mu_0}{2}}(1+i) = 26.05(1+i)$$

$$R = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}^*}{1+\tilde{\beta}^*}\right) = 0.926$$

Solutions provided by:
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