**NATIONAL UNIVERSITY OF SINGAPORE**

PC3231 Electricity and Magnetism 2

(Semester I: AY 2009-10)

Time Allowed: 2 Hours

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**INSTRUCTIONS TO CANDIDATES**

1. This examination paper contains **4** questions and comprises **4** printed pages.
2. Answer any **3** questions.
3. Answers to the questions are to be written in the answer books.
4. This is a CLOSED BOOK examination.
5. One Help Sheet (A4 size, both sides) is allowed for this examination.

1. Induced *emf* and forces

A copper ring of radius $a$ is at a fixed distance $d$ (with $a << d$) directly above an identical copper ring. Each ring has a resistance $R$ for circulating currents. An increasing current $I = I_0 t$ is applied in the lower ring. Neglect the self-inductance of each ring, and make suitable approximations.

(i) Determine the magnetic flux through the upper ring.

(ii) Find the induced *emf* and the current in the upper ring.

(iii) Show that the force $F$ between the rings is approximately

$$F \approx \frac{3\mu_0^2 \pi^2 a^8 I_0^2 t}{4Rd^7}$$

2.

(i) Gauge transformation

Show that in gauge transformations of Maxwell's equations in the potential formulation, we can add $\nabla\lambda$ to the vector potential **A** provided that we simultaneously subtract $\frac{\partial \lambda}{\partial t}$ from the electric potential $V$. Here $\lambda$ is a scalar function.

Use the gauge function $\lambda = -\frac{1}{4\pi\varepsilon_o}\frac{qt}{r}$ to transform the potentials.

$$V(\mathbf{r},t) = 0$$

$$\mathbf{A}(\mathbf{r},t) = -\frac{1}{4\pi\varepsilon_o}\frac{qt}{r^2}\hat{\mathbf{r}}$$

for a stationary point charge $q$.

(ii) Jefimenko's equation for E

Suppose the current density **J(r)** is constant in time, show that the Jefimenko's equation for **E** reduces to the usual Coulomb's law, that is,

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t)}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'$$

where the charge density is evaluated at the non-retarded time.

*Hint:* From the continuity equation, show firstly that the charge density $\rho$ is a linear function of time, such that

$$\rho(\mathbf{r},t) = \rho(\mathbf{r},0) + \dot{\rho}(\mathbf{r},0)t$$

where $\dot{\rho}(\mathbf{r},0)$ is the time derivative of $\rho$ at $t = 0$.

3. Rectangular waveguide

Consider the $TE_{10}$ mode of a rectangular waveguide propagating in the $z$ direction.

(i) What are the components ($E_x$, $E_y$) and ($B_x$, $B_y$, $B_z$) of the electric and magnetic fields for the $TE_{10}$ mode? You are given that

$$E_x = \frac{i}{\left(\omega/c\right)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

$$E_y = \frac{i}{\left(\omega/c\right)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$

$$E_z = 0$$

and

$$B_x = \frac{i}{\left(\omega/c\right)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

$$B_y = \frac{i}{\left(\omega/c\right)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

$$B_z = B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

(ii) Find the time averaged Poynting vector $\langle \mathbf{S} \rangle$ of the $TE_{10}$ mode in the waveguide.

(iii) Consider the X-band rectangular waveguide of cross-sectional dimensions 2.28 cm X 1.01 cm. Determine the group velocities of propagation for the first three TE modes in this waveguide when the driving frequency is 2 X $10^{10}$ Hz.

4 Skin depth and reflection of a conductor

Consider the case when an electromagnetic plane wave travelling in a non-conducting medium impinges on a conducting medium with conductivity σ at normal incidence. The plane wave solutions for electric and magnetic fields **E** and **B** in the conducting medium are

$$\mathbf{E} = \mathbf{E}_o e^{-k_- z} e^{i(k_+ z - \omega t)}$$

$$\mathbf{B} = \mathbf{B}_o e^{-k_- z} e^{i(k_+ z - \omega t)}$$

$$k_\pm = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} \pm 1\right]^{1/2}$$

(i) Show that the skin depth in a poor conductor is independent of frequency and that the skin depth in a good conductor is approximately $\lambda/2\pi$, where $\lambda$ is the wavelength in the conductor.

(ii) Derive an expression relating the reflected electric field to the incident electric field at normal incidence.

(iii) Estimate the intensity reflection coefficient for light at an air-to-gold interface at optical frequencies of $\omega = 5 \times 10^{15}$ /s. The conductivity $\sigma$ of gold is approximately $5 \times 10^7\ \Omega^{-1}\mathrm{m}^{-1}$.

***~ End of Paper ~***

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