Question 1 i)

Magnetic field,

$$\int \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$B(2\pi s) = \mu_0 I\left(\frac{\pi s^2}{\pi a^2}\right)$$

$$\vec{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}$$

Electric field, since w is very small,

$$\vec{E} = \frac{\sigma(t)}{\epsilon_0}\hat{z} = \frac{q(t)}{\pi\epsilon_0 a^2}\hat{z} = \frac{It}{\pi\epsilon_0 a^2}\hat{z}$$

Question 1 ii)

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)\left(\frac{It}{\pi\epsilon_0 a^2}\right)(\hat{z}\times\hat{\phi}) = \frac{I^2 st}{2\epsilon_0\pi^2 a^4}\hat{r}$$

$$u_{EM} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2$$

$$= \frac{1}{2}\epsilon_0\left(\frac{It}{\pi\epsilon_0 a^2}\right)^2 + \frac{1}{2\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)^2$$

$$= \frac{1}{2}\frac{I^2}{\pi^2 a^4}\left(\frac{t^2}{\epsilon_0} + \frac{\mu_0 s^2}{4}\right)$$

$$= \frac{1}{8}\frac{I^2}{\pi^2 a^4}(4c^2t^2 + s^2)$$

Question 1 iii)

$$\int u_{EM}\, d\tau = \int\int\int \frac{1}{8}\frac{I^2}{\pi^2 a^4}(4c^2t^2 + s^2)\, s\, ds\, d\phi\, dz$$

$$= \frac{1}{8}\frac{I^2}{\pi^2 a^4}\int_0^a 4c^2t^2 s + s^3\, ds \int_0^{2\pi} d\phi \int_{-\frac{w}{2}}^{\frac{w}{2}} dz$$

$$= \frac{1}{8}\frac{I^2}{\pi^2 a^4}\left[2c^2t^2s^2 + \frac{s^4}{4}\right]_0^a 2\pi w$$

$$= \frac{1}{4}\frac{I^2 w}{\pi a^2}\left(2c^2t^2 + \frac{a^2}{4}\right)$$

Question 1 iv)

$$\vec{\nabla}\cdot\vec{S} = -\frac{I^2 t}{\epsilon_0 \pi^2 a^4}, \qquad \frac{\partial u_{EM}}{\partial t} = \frac{I^2 t}{\pi^2 a^4 \epsilon_0}, \qquad \frac{\partial u_{mech}}{\partial t} = 0 \text{ since } \vec{J} = \rho\vec{v} = 0.$$

$$\therefore \vec{\nabla}\cdot\vec{S} = -\frac{\partial}{\partial t}(u_{EM} + u_{mech})$$

The total power flowing into the gap is equal to the rate of increase of energy in the gap.

Question 2 A) i)

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0} \qquad (1)$$

$$\left(\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J} \qquad (2)$$

Letting $\vec{A}' = \vec{A} + \vec{\nabla}\lambda, V' = V - \frac{\partial\lambda}{\partial t}$,

(1), $$\nabla^2 V' + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}') = \nabla^2\left(V - \frac{\partial\lambda}{\partial t}\right) + \frac{\partial}{\partial t}[\vec{\nabla}\cdot(\vec{A} + \vec{\nabla}\lambda)]$$

$$= \nabla^2 V - \frac{\partial}{\partial t}\nabla^2\lambda + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) + \frac{\partial}{\partial t}\nabla^2\lambda$$

$$= \nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})$$

Similarly,

(2), $$\left(\nabla^2\vec{A}' - \mu_0\epsilon_0\frac{\partial^2\vec{A}'}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}' + \mu_0\epsilon_0\frac{\partial V'}{\partial t}\right)$$

$$= \left[\nabla^2(\vec{A} + \vec{\nabla}\lambda) - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}(\vec{A} + \vec{\nabla}\lambda)\right] - \vec{\nabla}\left[\vec{\nabla}\cdot(\vec{A} + \vec{\nabla}\lambda) + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(V - \frac{\partial\lambda}{\partial t}\right)\right]$$

$$= \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) + \nabla^2\vec{\nabla}\lambda - \mu_0\epsilon_0\frac{\partial^2\vec{\nabla}\lambda}{\partial t^2} - \vec{\nabla}\nabla^2\lambda + \mu_0\epsilon_0\vec{\nabla}\frac{\partial^2\lambda}{\partial t^2}$$

$$= \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right)$$

$\therefore$ The gauge transformation of Maxwell's equations can be done by adding $\vec{\nabla}\lambda$ to $\vec{A}$ and subtracting $\frac{\partial\lambda}{\partial t}$ from V.

Question 2 A) ii)

For a point charge,

$$V' = -\frac{q}{4\pi\epsilon_0 r}, \qquad \vec{A}' = 0$$

We let λ to be

$$\lambda = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r}, \qquad \vec{\nabla}\lambda = \frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r}, \qquad \frac{\partial\lambda}{\partial t} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

$$\therefore V = V' - \frac{\partial\lambda}{\partial t} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r} + \frac{1}{4\pi\epsilon_0}\frac{q}{r} = 0$$

$$\vec{A} = \vec{A}' + \vec{\nabla}\lambda = \frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r}, \qquad \text{[shown]}$$

Question 2 A) iii)

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\right) = 0 = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$$

$\therefore$ It is a Lorentz Gauge and a Coulomb Gauge as well.

Question 2 B)

Since this is an iron sphere, the electric fields are

$$\vec{E}\begin{cases} 0, & r < R \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r}, & r > R \end{cases}$$

The angular momentum density,

$$\vec{\ell} = \epsilon_0\vec{r}\times(\vec{E}\times\vec{B})$$

$$= \epsilon_0\vec{E}(\vec{r}\cdot\vec{B}) - \epsilon_0\vec{B}(\vec{r}\cdot\vec{E})$$

$$= \epsilon_0\left(\frac{Q}{4\pi\epsilon_0 r^2}\hat{r}\right)\left(\frac{\mu_0}{4\pi}\frac{m}{r^2}2\cos\theta\right) - \epsilon_0\left[\frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})\right]\left(\frac{Q}{4\pi\epsilon_0 r}\right)$$

$$= -\frac{\mu_0 m Q}{16\pi^2 r^4}\sin\theta\,\hat{\theta}$$

$$= -\frac{\mu_0 m Q}{16\pi^2 r^4}\sin\theta\,(\cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z})$$

The angular momentum (only the $\hat{z}$ direction will survive),

$$\vec{L} = \int \ell\hat{z}\,d\tau$$

$$= \frac{\mu_0 m Q}{16\pi^2}\int_R^\infty \frac{1}{r^2}dr\int_0^\pi \sin^3\theta\,d\theta\int_0^{2\pi}d\phi$$

$$= \frac{\mu_0 m Q}{16\pi^2}\left[-\frac{1}{r}\right]_R^\infty \left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^\pi 2\pi$$

$$= \frac{\mu_0 m Q}{6\pi R}\hat{z} = \frac{2}{9}\mu_0 M Q R^2 \hat{z}$$

Question 3 i)

We let $\vec{E}$ propagate in the z-direction,

$$\vec{E} = \tilde{E}_0 e^{-k_-} e^{i(k_+ z - \omega t)}\hat{x}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & E_z \end{vmatrix} = -\frac{\partial\vec{B}}{\partial t}$$

$$\frac{\partial E_x}{\partial z}\hat{y} = -\frac{\partial\vec{B}}{\partial t}$$

$$\tilde{E}_0(-k_- + ik_+)e^{-k_-}e^{i(k_+ z - \omega t)}\hat{y} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{B} = \tilde{E}_0 \frac{k_+ + ik_-}{\omega} e^{-k_-}e^{i(k_+ z - \omega t)}\hat{y}$$

But we know that $\tilde{E}_0$ is complex, and it can be written as $\tilde{E}_0 = E_0 e^{i\delta_E}$. So

$$\tilde{B}_0 = B_0 e^{i\delta_B} = \tilde{E}\frac{k_+ + ik_-}{\omega} = E_0 e^{i\delta_E}\frac{K}{\omega}e^{i\phi} = \frac{E_0 K}{\omega}e^{i(\delta_B + \phi)}$$

∴ We see that $\delta_B = \delta_E + \phi$, and they are not in phase. So $\vec{B}$ lags behind $\vec{E}$.

Question 3 ii)

We were given $\vec{E}_T = \tilde{E}_{0T}e^{-k_-}e^{i(k_+ z - \omega t)}\hat{x}$, in the medium (transmitted). Outside we have the incident wave, $\vec{E}_I = \tilde{E}_{0I}e^{i(k_1 z - \omega t)}\hat{x}$ and the reflected wave $\vec{E}_R = \tilde{E}_{0R}e^{i(-k_1 z - \omega t)}\hat{x}$.

The boundary conditions give

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}, \qquad \tilde{E}_{0I} - \tilde{E}_{0R} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}\tilde{E}_{0T} = \tilde{\beta}\tilde{E}_{0T}$$

Solving both equations, we get

$$\tilde{E}_{0R} = \frac{1-\tilde{\beta}}{2}\tilde{E}_{0I}$$

$$\therefore \vec{E}_R = \frac{1-\tilde{\beta}}{2}\tilde{E}_{0I}e^{i(-k_1 z - \omega t)}\hat{x}$$

Question 3 iii)

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2 = \frac{c}{\omega}(k_+ + ik_-) \approx \frac{c}{\omega}\sqrt{\frac{\sigma\omega\mu_0}{2}}(1+i) = 26.05(1+i)$$

$$R = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}^*}{1+\tilde{\beta}^*}\right) = 0.926$$

Question 4 i)

For $v \ll c, \vec{u} \approx c\hat{r}$,

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{r}{(c\hat{r}\cdot\vec{r})^3}[c^2 c\hat{r} + \vec{r}\times(c\hat{r}\times\vec{a})]$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{c^3 r^2}\{c^3\hat{r} + cr[\hat{r}\times(\hat{r}\times\vec{a})]\}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{c^3 r^2}\{c^3\hat{r} + cr[(\vec{a}\cdot\hat{r})\hat{r} - \vec{a}]\}$$

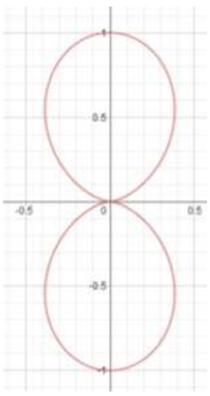
For the Poynting vector, only the radiation field contributes, so

$$\vec{E}_{rad}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{c^2 r}[(\vec{a}\cdot\hat{r})\hat{r} - \vec{a}] = \frac{\mu_0 q}{4\pi r}[(\vec{a}\cdot\hat{r})\hat{r} - \vec{a}]$$

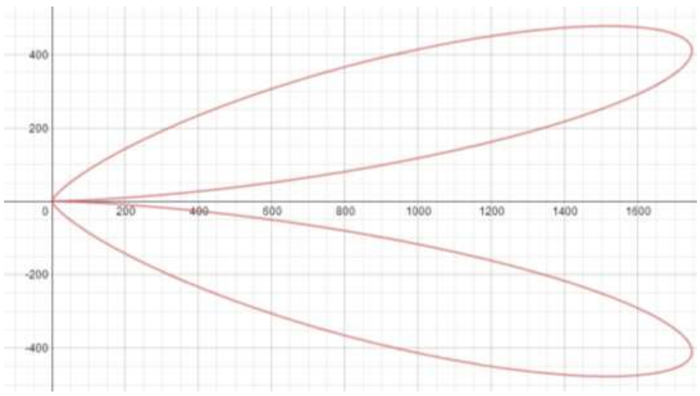
$$\therefore \vec{S}_{rad} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{\mu_0}\left[\vec{E}\times\frac{1}{c}(\hat{r}\times\vec{E})\right] = \frac{1}{c\mu_0}[E^2\hat{r} - (\vec{E}\cdot\hat{r})\vec{E}] = \frac{1}{c\mu_0}\left(\frac{\mu_0 q}{4\pi r}\right)^2[a^2 - (\vec{a}\cdot\hat{r})^2]$$

Question 4 ii)

$v \ll c$,



$v \approx c$,



Question 4 iii)

$$P = \int S\, d\Omega$$

$$= \frac{\mu_0 q^2}{16\pi^2 c}\int \frac{a^2 - (\vec{a}\cdot\hat{r})^2}{r^2} r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{\mu_0 q^2}{8\pi c}\int (a^2 - a^2\cos^2\theta)\sin\theta\, d\theta$$

$$= \frac{\mu_0 q^2 a^2}{8\pi c} \int \sin^3 \theta \, d\theta$$

$$= \frac{\mu_0 q^2 a^2}{8\pi c} \left(\frac{4}{3}\right)$$

$$= \frac{\mu_0 q^2 a^2}{6\pi c}$$

Question 4 iv)

The initial kinetic energy,

$$T = \frac{1}{2} m_e v_0^2$$

Power loss,

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

Energy loss,

$$E = \frac{\mu_0 q^2 a^2}{6\pi c} \left(\frac{v_0}{a}\right) = \frac{\mu_0 q^2 a v_0}{6\pi c}$$

$\therefore$ The fraction loss of energy $= \frac{E}{T} = \frac{\mu_0 q^2 a}{3\pi c m_e v_0}$

Solutions provided by:

D.J. Griffiths (Q1i-ii, 2Ai-ii, 2B, 3i, 3iii, 4i-iii)

John Soo (Q1iii-iv, 2Aiii, 3ii, 4iv)