Question 1 (a)

$$\int_0^2 \frac{\sinh x}{x}\,dx$$

We use the formula $T_{a,b} = \dfrac{4^{b-1}T_{a,b-1} - T_{a-1,b-1}}{4^{b-1}-1}$ to construct the triangle below:

h	$T_{a,b}$		
h	T_{11}		
$\frac{h}{2}$	T_{21}	T_{22}	
$\frac{h}{4}$	T_{31}	T_{32}	T_{33}

… … … … …

Here T_{aa} is the integral calculated using the Composite Trapezoidal Rule,

$T_{11} = \frac{h}{2}[f(2) - f(0)]$, $\quad T_{22} = \frac{h}{4}[f(2) + 2f(1) + f(0)]$, $\quad$ etc.

So after a lengthy calculation, we get

h	$T_{a,b}$				
2	2.81343				
1	2.58192	2.50475			
0.5	2.52182	2.51781	2.51868		
0.25	2.50664	2.50640	2.50622	2.50602	
0.125	2.50284	2.50283	2.50282	2.50281	2.50280

Our final answer obtained is 2.5028 (exact answer: 2.5016).

Tips: It is best to further attempt one more line to get a more accurate answer, but the calculations will involve at least 17 terms!

Question 1 (b)

$f(x) = 3x^3 - 10x^2 + 5x + 5$

The root is at (2,3).

$x_1 = 2$, $\quad x_2 = 3$, $\quad x_3 = 2.5$,

$f_1 = -1$, $\quad f_2 = 11$, $\quad f_3 = \frac{15}{8}$

$$x_{(0)} = -\frac{f_2 f_3 x_1 (f_2 - f_3) + f_3 f_1 x_2 (f_3 - f_1) + f_1 f_2 x_3 (f_1 - f_2)}{(f_1 - f_2)(f_2 - f_3)(f_3 - f_1)}$$

$$= -\frac{\frac{12045}{32} - \frac{1035}{64} + \frac{225}{4}}{-\frac{5037}{16}}$$

$$= 2.192525$$

Inside bracket, so

$x_1 = 2, \quad x_2 = 2.5, \quad x_3 = 2.192525$

$f_1 = -1, \quad f_2 = \frac{15}{8}, \quad f_3 = -0.489538$

$x_{(1)} = 2.335080$

Repeating the process, we get

$x_{(2)} = 2.280523$

$x_{(3)} = 2.284426$

$x_{(4)} = 2.284324$

$x_{(5)} = 2.284324$

$\therefore x = 2.28432 \ (5\ d.p.)$

Tips: To speed up the process, we can also alternate between polynomial interpolation and bisection. Although the question specifies a specific interpolation, the bisection method is allowed to be used in between.

Question 1 (c)

x	y					
1	-2					
-1	-14	6				
3	18	10	1			
2	1	3	-1	2		
4	61	21	3	2	0	
-2	-47	15	-9	2	0	0

$y = -2 + 6(x+1) + (x^2 - 1) + 2(x^2 - 1)(x - 3) = -3 + 4x - 5x^2 + 2x^3$

Question 2 (a)

$$\sqrt{x+1}\frac{dy}{dx} + e^{-x}y^3 = 0$$

$$\frac{dy}{dx} = -\frac{e^{-x}y^3}{\sqrt{x+1}}$$

We use Heun's method, $y_1 = y_0 + \frac{h}{2}(f_0 + f_1)$

where $f_0 = f(x_0, y_0)$, $f_1 = f(x_0 + h, y_0 + hf_0)$. So we get

$$y_1 = y_0 + \frac{h}{2}\left[\frac{e^{-x_0}y_0^3}{\sqrt{x_0+1}} + \frac{e^{-x_0+h}(y_0 + hf_0)^3}{\sqrt{x_0+h+1}}\right]$$

Plug $x_0 = 0, y_0 = 2$ into the equation, we get $y_1 = 1.52546$, $x_1 = 0.1$. Repeating the process, we will get

$x_2 = 0.2,\quad y_2 = 1.30461$
$x_3 = 0.3,\quad y_3 = 1.17367$
$x_4 = 0.4,\quad y_4 = 1.08658$
$x_5 = 0.5,\quad y_5 = 1.02443$
$x_6 = 0.6,\quad y_6 = 0.97794$
$x_7 = 0.7,\quad y_7 = 0.94197$
$x_8 = 0.8,\quad y_8 = 0.913425$
$x_9 = 0.9,\quad y_9 = 0.89034$
$x_{10} = 1.0,\quad y_{10} = 0.87138$

Question 2 (b)

x	1	1.5	1.8	2.4
$f(x)$	6	6.875	7.952	12.104

We first do an interpolation (your choice of method), then the Gaussian quadrature to evaluate the integral.

$$x = \frac{b+a}{2} + \frac{b-a}{2}\xi = 1.7 + 0.5\xi$$

$$\xi = \pm\frac{1}{\sqrt{3}},\qquad w = 1$$

$$\int_{1.2}^{2.2} f(x)\,dx = \frac{1}{2}\left[f\left(1.7 - \frac{1}{2\sqrt{3}}\right) + f\left(1.7 + \frac{1}{2\sqrt{3}}\right)\right] = \frac{1}{2}[f(1.41132) + f(1.98868)] = 7.79134$$

Tips: you can also use Newton's interpolating polynomial, which yields the integral

$$\int_{1.2}^{2.2} x^3 - 2x^2 + 2x + 5\,dx.$$

Question 3 (a)

$h = 0.1, \qquad \tau = 0.1$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{h^2} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\tau^2}$$

$$u_j^{n+1} = u_{j+1}^n + u_{j-1}^n - u_j^{n-1}, \qquad (1)$$

We want to find $u(x, 0.2)$, $u_j^2 = u_{j+1}^1 + u_{j-1}^1 - u_j^0$.

Since the ends are fixed, we have $u_0^2 = u_0^1 = 0$, $u_{10}^2 = u_{10}^1 = 0$.

The information given were $u_0^0 = 0$, $u_j^0 = e^{-100(x_j - 0.5)^2}$, $u_{10}^0 = 0$.

It is initially not moving, so $\dfrac{\partial u_j^0}{\partial t} = \dfrac{u_j^1 - u_j^{-1}}{2\tau} = 0 \quad \Rightarrow \quad u_j^{-1} = u_j^1$.

Now using equation (1) and the information obtained, we get

$$u_j^1 = u_{j+1}^0 + u_{j-1}^0 - u_j^{-1} \quad \Rightarrow \quad u_j^1 = \frac{u_{j+1}^0 + u_{j-1}^0}{2}$$

Use this new information to find a relationship between u^2 and u^0:

$$u_j^2 = u_{j+1}^1 + u_{j-1}^1 - u_j^0 = \frac{u_{j+2}^0 + u_{j-2}^0}{2}$$

$$u_0^2 = 0, \qquad u_1^2 = \frac{u_3^0 + u_{-1}^0}{2} = 0.00916$$

Continue calculating, we get

$u_2^2 = 0.18394$

$u_3^2 = 0.50000$

$u_4^2 = 0.18400$

$u_5^2 = 0.01832$

$u_6^2 = 0.18400$

$u_7^2 = 0.50000$

$u_8^2 = 0.18394$

$u_9^2 = 0.00916$

$u_{10}^2 = 0$

Question 3 (b)

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_j^{n+1} - u_j^n}{\tau} = \alpha \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}$$

$$u_j^{n+1} = u_j^n + \frac{\alpha\tau}{h^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

We let

$$u_j^{n+1} = \xi u_j^n, \qquad x_j = jh, \qquad u_j^n = A^n e^{hijk}$$

$$A^{2n+1} e^{hijk} = A^n e^{hijk} + \frac{\alpha\tau}{h^2}\left(A^n e^{hi(j+1)k} - 2A^n e^{hijk} + A^n e^{hi(j-1)k}\right)$$

$$\xi = 1 + \frac{\tau\alpha}{h^2}\left(e^{ikh} + e^{-ikh} - 2\right) = 1 - \frac{4\tau\alpha}{h^2}\sin^2\frac{kh}{2}$$

$$|\xi| \le 1$$

$$\left|1 - \frac{4\tau\alpha}{h^2}\right| \le 1$$

$$0 \le \frac{4\tau\alpha}{h^2} \le 2$$

∴ For it to be stable, $\frac{\tau\alpha}{h^2} = \frac{1}{2}$.

Question 4

$$(x-2)\frac{d^2y}{dx^2} - 6\sin x^2 \frac{dy}{dx} + y(1-x^2)\cos x = 0$$

$$y'' = \frac{6\sin x^2}{x-2}y' - \frac{(1-x^2)\cos x}{x-2}y$$

We let $y_1 = y$, $y_2 = y'$, we have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{6\sin x^2}{x-2}y_2 - \frac{(1-x^2)\cos x}{x-2}y_1 \end{pmatrix}, \qquad y_{2,0} = -5, \qquad y_{2,10} = 2$$

Using the simple Euler Method,

$$y_{1,n+1} = y_{1,n} + \frac{1}{10}y_{2,n}$$

$$y_{2,n+1} = y_{2,n} + \frac{1}{10}\frac{6\sin x_n^2}{x_n - 2}y_2 - \frac{(1-x_n^2)\cos x_n}{x_n - 2}y_1$$

Using a trial value of $y_{1,0} = 1$,

$y_{1,0} = 1, \quad y_{2,0} = -5$

$y_{1,1} = 0.5, \quad y_{2,1} = -4.95$

$y_{1,2} = 0.005, \quad y_{2,2} = -4.90845$

$y_{1,3} = -0.48585, \quad y_{2,3} = -4.84276$

$y_{1,4} = -0.97013, \quad y_{2,4} = -4.71398$

$y_{1,5} = -1.44153, \quad y_{2,5} = -4.47926$

$y_{1,6} = -1.88946, \quad y_{2,6} = -4.09924$

$y_{1,7} = -2.29938, \quad y_{2,7} = -3.55165$

$y_{1,8} = -2.65455, \quad y_{2,8} = -2.84918$

$y_{1,9} = -2.93947, \quad y_{2,9} = -2.05390$

$y_{1,10} = -3.14486, \quad y_{2,10} = -1.27404$

The residue function, $r(1) = -1.27404 - 2 = -3.27404$

Now using a different trial value $y_{1,0} = 2$, and repeating the calculations, we get

$y_{1,10} = -1.97561, \quad y_{2,10} = -1.14655$

The residue function, $r(2) = -1.14655 - 2 = -3.14655$

Since the residue function is linear, we have

$$r(y_{1,0}) = \frac{-3.14655 + 3.27404}{2 - 1}(y_{1,0} - 1) - 3.27404 = 0.12749y_{1,0} - 3.40153$$

$\therefore r = 0, y_{1,0} = 26.68076$

Using this new value of $y_{1,0}$, we get

$y_{1,0} = 26.68076, \quad y_{2,0} = -5$

$y_{1,1} = 26.18076, \quad y_{2,1} = -3.66596$

$y_{1,2} = 25.81416, \quad y_{2,2} = -2.29705$

$y_{1,3} = 25.58446, \quad y_{2,3} = -0.91711$

$y_{1,4} = 25.49275, \quad y_{2,4} = 0.42033$

$y_{1,5} = 25.53478, \quad y_{2,5} = 1.62794$

$y_{1,6} = 25.69758, \quad y_{2,6} = 2.58728$

$y_{1,7} = 25.95631, \quad y_{2,7} = 3.16623$

$y_{1,8} = 26.27293, \quad y_{2,8} = 3.25731$

$y_{1,9} = 26.59866, \quad y_{2,9} = 2.83382$

$y_{1,10} = 26.88204, \quad y_{2,10} = 1.99986$

Tips: The simple Euler method does not produce accurate results. The results calculated using the 4th order Runge-Kutta method yields

$r(1) = -1.21141 - 2 = -3.21141$

$r(2) = -1.10362 - 2 = -3.10362$

$y_{1,0} = 30.79380$

Solutions provided by:

A/Prof Paul Lim (Questions 1b – 2, 3b)

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