The answers for certain questions in this document are incomplete. Would you like to help us complete them? If yes, Please send your suggested answers to nus.physoc@gmail.com. Thanks!

Question 1 (a)

Question 1 (b)

$f_1 = \tan x - \cos y - 1$

$f_2 = \cos x + 3\sin y$

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \sec^2 x & \sin y \\ -\sin x & 3\cos y \end{pmatrix}$$

$$\Delta x = -J^{-1} f = -\frac{1}{3\sec^2 x \cos y + \sin x \sin y} \begin{pmatrix} 3\cos y & -\sin y \\ \sin x & \sec^2 x \end{pmatrix} \begin{pmatrix} \tan x - \cos y - 1 \\ \cos x + 3\sin y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}_0 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}_1 = \begin{pmatrix} x \\ y \end{pmatrix}_0 + \Delta x|_{3,3} = \begin{pmatrix} 3.17707 \\ 2.8008 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}_2 = \begin{pmatrix} 3.19858 \\ 2.80225 \end{pmatrix}$$

Question 2

Question 3

$$\frac{d^2y}{dx^2} + 2000x^3y = 0,$$

$$y(0) = 0, \qquad y(0.3) = 1, \qquad h = 0.1$$

$$y_1 = y_0 + hf(x_{mid}, y_{mid})$$

Writing the equation in vector form,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ -2000x^3y_1 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{mid} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_0 + \frac{h}{2}\begin{pmatrix} y_2 \\ -2000x^3y_1 \end{pmatrix}_0$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_0 + h\begin{pmatrix} y_{2,mid} \\ -2000x^3 y_{1,mid} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_0 + h\begin{pmatrix} y_{2,0} - 1000hx^3 y_{1,0} \\ -2000\left(x_0 + \frac{h}{2}\right)^3 \left(y_{1,0} + \frac{h}{2} y_{2,0}\right) \end{pmatrix}$$

Using the initial values, $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we get

$x_1 = 0.1,\quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_1 = \begin{pmatrix} 0.1 \\ 0.99875 \end{pmatrix}$

$x_2 = 0.2,\quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_2 = \begin{pmatrix} 0.198875 \\ 0.897542 \end{pmatrix}$

$x_3 = 0.3,\quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_3 = \begin{pmatrix} 0.2727192 \\ 0.1358166 \end{pmatrix}$

The residue function obtained,

$r = y_1|_{x=0.3} - 1 = -0.7272808$

Now we choose another initial value, $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_0 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, then we get $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_3 = \begin{pmatrix} 0.545438 \\ 0.27163368 \end{pmatrix}$ and $r = -0.454562$.

Since the residue function is a straight line, we can use some interpolation (e.g.Lagrange) to find the value of u, which is $u = 3.66678$. Finally we verify the answer:

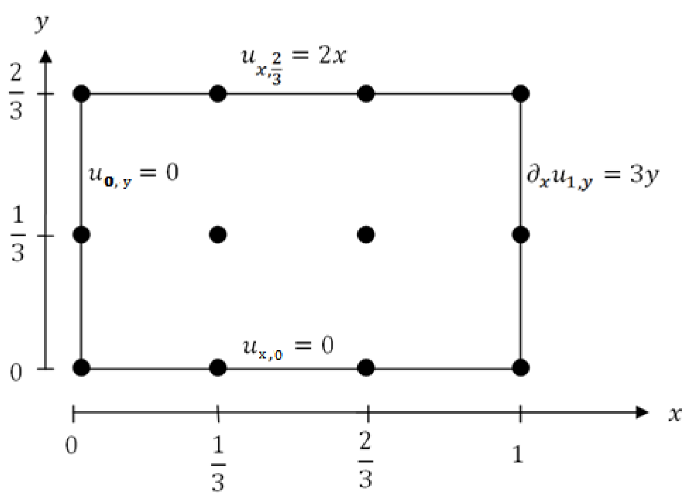
$x_0 = 0,\quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_0 = \begin{pmatrix} 0 \\ 3.66678 \end{pmatrix}$

$x_1 = 0.1,\quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_1 = \begin{pmatrix} 0.366678 \\ 3.6621965 \end{pmatrix}$

$x_2 = 0.2,\quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_2 = \begin{pmatrix} 0.72923087 \\ 3.29109 \end{pmatrix}$

$x_3 = 0.3,\quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_3 = \begin{pmatrix} 1 \\ 0.49801 \end{pmatrix}$

Question 4



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The value of u at one point is the average between the 4 adjacent points. So listing $u_{x,y}$ from the bottom to the top,

$$u_{0,0} = u_{\frac{1}{3},0} = u_{\frac{2}{3},0} = u_{1,0} = 0$$

$$u_{0,\frac{1}{3}} = 0$$

$$u_{\frac{1}{3},\frac{1}{3}} = \frac{1}{4}\left(u_{\frac{1}{3},0} + u_{0,\frac{1}{3}} + u_{\frac{2}{3},\frac{1}{3}} + u_{\frac{1}{3},\frac{2}{3}}\right) = \frac{1}{4}\left(u_{\frac{2}{3},\frac{1}{3}} + \frac{2}{3}\right), \qquad (1)$$

$$u_{\frac{2}{3},\frac{1}{3}} = \frac{1}{4}\left(u_{\frac{1}{3},\frac{1}{3}} + u_{1,\frac{1}{3}} + \frac{4}{3}\right), \qquad (2)$$

$$u_{1,\frac{1}{3}} = \frac{1}{4}\left(u_{\frac{2}{3},\frac{1}{3}} + u_{\frac{4}{3},\frac{1}{3}} + 2\right), \qquad (3)$$

$$u_{0,\frac{2}{3}} = 0$$

$$u_{\frac{1}{3},\frac{2}{3}} = \frac{2}{3}$$

$$u_{\frac{2}{3},\frac{2}{3}} = \frac{4}{3}$$

$$u_{1,\frac{2}{3}} = 2$$

From the condition $\frac{\partial u_{1,y}}{\partial x} = 3y$, we get

$$\frac{1}{2\left(\frac{1}{3}\right)}\left(u_{\frac{4}{3},\frac{1}{3}} - u_{\frac{2}{3},\frac{1}{3}}\right) = 1, \qquad (4)$$

Solving (1) to (4), we get

$$u_{\frac{1}{3},\frac{1}{3}} = \frac{1}{3}, \qquad u_{\frac{2}{3},\frac{1}{3}} = \frac{2}{3}, \qquad u_{1,\frac{1}{3}} = 1.$$

Solutions Provided by:

A/Prof Paul Lim (Questions 1b, 3 – 4)

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