NATIONAL UNIVERSITY OF SINGAPORE

PC3238 FLUID DYNAMICS

(Semester 2: AY 2012–13)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains FOUR (4) questions and comprises SIX (6) printed pages, inclusive of this cover page.
2. Answer any THREE (3) questions.
3. Answers to the questions are to be written in the answer books.
4. This is a CLOSED BOOK examination.
5. One Help Sheet (A4 size, both sides) is allowed for this examination.

1. EITHER

For 2-D irrotational flows of an incompressible and non-viscous fluid, the flow field $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$ may be written in terms of a complex potential $w(z)$:

$$dw/dz = u - iv, \quad \text{with} \quad w(z) = \phi(x,y) - i\psi(x,y), \quad \text{where} \quad z = x + iy.$$

(a) Show that the net pressure force components F_x and F_y on a 2-D object immersed in the flow and the moment M about the origin may be evaluated by the following contour integrals around the object (Blasius Theorem):

$$F_x - iF_y = \frac{i\rho}{2} \oint \left(\frac{dw}{dz}\right)^2 dz,$$

and

$$M = \text{Real}\left[-\frac{\rho}{2} \oint z \left(\frac{dw}{dz}\right)^2 dz\right],$$

where ρ is the density of the fluid.

(b) By the following transforms:

$$z_1 = z_0\, \mathrm{e}^{i\delta}, \quad z_2 = z_1 + \frac{b^2}{z_1}, \quad \text{and} \quad z = z_2\, \mathrm{e}^{-i\delta},$$

the flow $w(z_0)$ around a circle of radius a $(a > b)$ in the z_0-plane:

$$w(z_0) = U z_0 + \frac{Ua^2}{z_0}$$

is transformed to a flow $w_1(z) = w\Big\{z_0\Big[z_1\Big(z_2(z)\Big)\Big]\Big\}$ around an ellipse at an angle of attack δ in the z-plane.

Determine the lift, drag, and moment experienced by the ellipse.

The following equations might be useful:

$$\oint f(z)\, dz = 2\pi i \sum \textit{Residues}.$$

For $|z| < |\beta|$,

$$\frac{(z^2-\alpha^2)^2}{(z^2-\beta^2)} = -\frac{\alpha^4}{\beta^2} + \left(\frac{2\alpha^2}{\beta^2} - \frac{\alpha^4}{\beta^4}\right) z^2 + \left(-\frac{1}{\beta^2} + \frac{2\alpha^2}{\beta^4} - \frac{\alpha^4}{\beta^6}\right) z^4 + \cdots;$$

$$\frac{(z^2-\alpha^2)^2(z^2+\beta^2)}{(z^2-\beta^2)} = -\alpha^4 + 2\alpha^2\left(1 - \frac{\alpha^2}{\beta^2}\right) z^2 + \cdots.$$

OR

A sphere of radius a is moving with a velocity $U(t)$ in a fixed direction through an incompressible, non-viscous fluid.

(a) Prove that the three-dimensional flow of the fluid caused by the moving sphere may be given in terms of the flow potential

$$\phi(r,\theta,t) = -\frac{U(t)a^3}{2r^2}\cos\theta,$$

where r is the radial distance from the centre of the sphere, θ is the angle between the radial vector and the direction of motion.

Note: The Laplacian in spherical coordinates is

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

where φ is the azimuthal angle.

(b) Show that the kinetic energy of the flow caused by the moving sphere at time t turns out to be exactly equal to

$$E_f = \frac{1}{4}M_f[U(t)]^2$$

where M_f is the mass of the fluid displaced by the sphere.

(c) A force is applied on the sphere to maintain a constant acceleration A such that $U(t) = At$. Show that the force has to be of a magnitude

$$F = \left(M_s + \frac{M_f}{2}\right)A$$

where M_s is the mass of the sphere and M_f is the mass of the fluid displaced by the sphere.

The Bernoulli's equation for an evolving incompressible, irrotational flow is

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} = \text{constant}.$$

2. The shallow water equations in one-dimension are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x},$$
$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = -h\frac{\partial u}{\partial x}.$$

(a) Derive the characteristics equations

$$\frac{\partial(u+2c)}{\partial t} + (u+c)\frac{\partial(u+2c)}{\partial x} = 0,$$
$$\frac{\partial(u-2c)}{\partial t} + (u-c)\frac{\partial(u-2c)}{\partial x} = 0.$$

where $c = \sqrt{gh}$.

(b) Consider a simple wave propagating in the positive x-direction into still water with a constant depth of h_0. Show that the equation for the fluid depth may be written as

$$\frac{\partial h}{\partial t} + \left(3\sqrt{gh} - 2\sqrt{gh_0}\right)\frac{\partial h}{\partial x} = 0.$$

Suppose $h = h_0 + \eta$, with $|\eta| \ll h_0$, derive the following equation for the fluid depth perturbation η:

$$\frac{\partial \eta}{\partial t} + c_0\left(1 + \frac{3}{2}\frac{\eta}{h_0}\right)\frac{\partial \eta}{\partial x} = 0, \qquad \text{where} \quad c_0 = \sqrt{gh_0}.$$

(c) The dispersion relation of water waves is given by

$$\omega^2 = gk\tanh(kh_0).$$

Show that for shallow water wave propagating in the positive x-direction,

$$\omega \approx c_0 k - \frac{c_0}{6}h_0^2 k^3.$$

(d) Show that the following equation is appropriate for study of mildly nonlinear shallow water waves with the correct dispersion characteristics:

$$\frac{\partial \eta}{\partial t} + c_0\left(1 + \frac{3}{2}\frac{\eta}{h_0}\right)\frac{\partial \eta}{\partial x} + \frac{1}{6}c_0 h_0^2\frac{\partial^3 \eta}{\partial x^3} = 0.$$

The following equations might be useful:

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}};$$
$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots.$$

3. Consider the parallel flow of two layers of incompressible, inviscid fluids. The fluid layer above ($z > 0$) has density ρ_1 and flow speed U_1 in the x-direction. The fluid layer below ($z < 0$) has density ρ_2 and flow speed U_2 in the same direction.

The linearised equations for small-amplitude 2-D perturbations in the x-z-plane of such a two-fluid system are:

$$\nabla^2 \phi_i = 0, \quad i = 1 \text{ for } z > 0 \text{ and } i = 2 \text{ for } z < 0, \quad \phi_i \to 0 \text{ as } |z| \to \infty;$$

$$\frac{\partial \phi_i}{\partial z} = \frac{\partial \eta}{\partial t} + U_i \frac{\partial \eta}{\partial x}, \quad \text{at } z = 0 \text{ for } i = 1, 2;$$

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} + g\eta \right) + T \frac{\partial^2 \eta}{\partial x^2} = \rho_2 \left(\frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x} + g\eta \right), \quad \text{at } z = 0.$$

Note that the effect of surface tension is represented by the term $T\partial^2\eta/\partial x^2$.

(a) Consider a sinusoidal mode with $\phi_1 = Ae^{st-ikx-kz}$, $\phi_2 = Be^{st-ikx+kz}$ and $\eta = Ce^{st-ikx}$, derive the following expression for the growth rate s:

$$s = \frac{ik(\rho_1 U_1 + \rho_2 U_2)}{\rho_1 + \rho_2} \pm \frac{\sqrt{k^2 \rho_1 \rho_2 (U_1 - U_2)^2 - kg(\rho_2^2 - \rho_1^2) - k^3(\rho_1 + \rho_2)T}}{\rho_1 + \rho_2}.$$

(b) Show that if $\rho_1 = 0$ and $U_1 = U_2 = 0$, the standard dispersion relation for deep water surface waves is obtained:

$$\omega^2 = gk + Tk^3/\rho_2.$$

(c) Show that if $T = 0$ and $U_1 \neq U_2$, the flow is always unstable to perturbations of sufficiently large wavenumber, viz., those with

$$k > \frac{g(\rho_2^2 - \rho_1^2)}{\rho_1 \rho_2 (U_1 - U_2)^2}.$$

(d) Show that surface tension will help to stabilise the flow, and the flow will be stable provided that

$$(U_1 - U_2)^2 < \frac{2}{\rho_1 \rho_2} \sqrt{gT(\rho_2 - \rho_1)(\rho_1 + \rho_2)^2}.$$

4. (a) Consider the flow of a body of incompressible fluid governed by the Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0.$$

Derive the following equation for the local rate of change of the kinetic energy density:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}^2\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho\mathbf{u}^2 + p\right)\mathbf{u} + \mu\mathbf{w}\times\mathbf{u}\right] = -\mu\mathbf{w}^2$$

where $\mathbf{w} = \nabla\times\mathbf{u}$ and $\mu = \rho\nu$.

(b) Write a brief note on Kolmogorov's 1941 theory for fully developed, isotropic turbulence, including the arguments leading to the following conclusions:

- The length scale η and the velocity scale v of the smallest eddies are related to the generation length scale ℓ and velocity scale u as

$$\eta \sim Re^{-3/4}\ell, \qquad v \sim Re^{-1/4}u.$$

- The inertial subrange energy spectrum $E(k)$ is expected to depend on the $[-5/3]$-power of the wavenumber k:

$$E(k) \sim \epsilon^{2/3}k^{-5/3},$$

where ϵ is the rate of energy cascade.

The following equations might be useful:

$$\mathbf{u}\cdot\nabla\varphi = \nabla(\varphi\mathbf{u}) - \varphi\nabla\cdot\mathbf{u};$$

$$\mathbf{u}\cdot\nabla\mathbf{u} = (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla\frac{1}{2}\mathbf{u}^2;$$

$$\nabla\cdot(\mathbf{u}\times\mathbf{w}) = (\nabla\times\mathbf{u})\cdot\mathbf{w} - (\nabla\times\mathbf{w})\cdot\mathbf{u};$$

$$\nabla\times(\nabla\times\mathbf{u}) = \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u}.$$

(LH)

– End of Paper –