NATIONAL UNIVERSITY OF SINGAPORE

PC3238 – FLUID DYNAMICS

(Semester 1: AY 2013–14)

Name of Examiner: Prof Lim Hock

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name.**
2. This examination paper contains FOUR (4) questions and comprises FIVE (5) printed pages, inclusive of this cover page.
3. Answer any THREE (3) questions.
4. Answers to the questions are to be written in the answer books.
5. Please start each question on a new page.
6. This is a CLOSED BOOK examination.
7. One Help Sheet (A4 size, both sides) is allowed for this examination.

1. An object S with impermeable surface is moving in an infinite body of non-viscous, incompressible fluid. Consider the fluid within the control volume V bounded internally by the surface of S and externally by a very large, fixed (imaginary) sphere S_R of radius R. The fluid flow generated may be assumed to be irrotational.

   (a) Show that the rate of change of $\mathbf{K}$, the total momentum of the fluid in V, may be written as:

$$\frac{d\mathbf{K}}{dt} = \oint_{S_R} \rho \frac{\partial \phi}{\partial t} \mathbf{n}\, dS + \frac{d}{dt} \oint_S \rho \phi \mathbf{n}\, dS.$$

   (b) Explain why the time derivative of the second term cannot be moved into the integral as in the first term.

   (c) Show that the net pressure force acting on the volume of fluid through the surface S_R is

$$\mathbf{F}_{S_R} = \oint_{S_R} (-p)\mathbf{n}\, dS = \oint_{S_R} \rho \frac{\partial \phi}{\partial t} \mathbf{n}\, dS + \oint_{S_R} \frac{1}{2} \rho (\nabla \phi)^2 \mathbf{n}\, dS.$$

   (d) Show that the momentum flux per unit time flowing out of the volume through S_R is

$$\mathbf{f}_{S_R} = \oint_{S_R} (\rho \nabla \phi) \mathbf{n} \cdot \nabla \phi\, dS.$$

   (e) Explain why the momentum change may be described by the following equation

$$\frac{d\mathbf{K}}{dt} = \mathbf{F}_{S_R} - \mathbf{f}_{S_R} + \mathbf{F}_S$$

   where $\mathbf{F}_S$ is the pressure force exerted by the object S on the fluid.

   (f) Derive the following equation for $\mathbf{F}_S$:

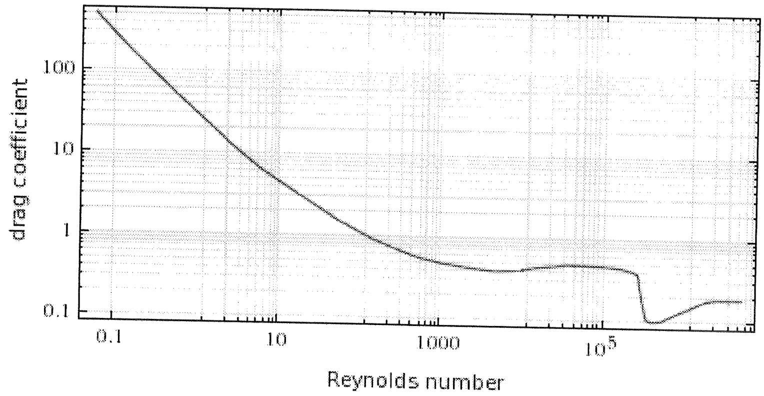
$$\mathbf{F}_S = \frac{d}{dt} \oint_S \rho \phi \mathbf{n}\, dS.$$

   (g) Explain why the term $\oint_S \rho \phi \mathbf{n}\, dS$ is sometimes called the impulse.

The following equations might be useful:

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\mathbf{v}^2 = \text{constant}, \qquad \int_V \nabla \psi\, dV = \oint_S \psi \mathbf{n}\, dS.$$

2. The figure below is a log-log plot of Drag Coefficient C_D versus Reynolds Number Re for viscous flow around a sphere.



(a) Consider a viscous fluid of density ρ and viscosity μ flowing around a sphere of radius a. The flow speed far from the sphere is U. Perform a dimensional analysis to show that the drag D experienced by the sphere is expected to depend on the various parameters in the following manner:

$$\frac{D}{\rho U^2 a^2} = \Phi\left(\frac{\rho a U}{\mu}\right),$$

where $\Phi(x)$ is a general function of the argument x.

(b) Briefly state the relevance of the above equation to the plot of Drag Coefficient versus Reynolds Number shown above.

(c) The Stokes formula gives the drag on a sphere as $D = 6\pi\mu a U$. What is the function $\Phi(x)$ introduced in Part (a) when the Stokes formula is valid?

(d) In which part of the above plot of Drag Coefficient versus Reynolds Number is the Stokes formula applicable, and what should be the gradient of the plot when the Stokes formula is valid?

(e) State the range of values of the Reynolds Number where the drag is approximately proportional to U^2.

(f) A sharp decrease in the Drag Coefficient is observed at Reynolds Number around 1.5×10^5. Briefly describe the physical phenomena that cause the sharp drop in drag.

3. The shallow water equations in one-dimension are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}, \qquad \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = -h\frac{\partial u}{\partial x}.$$

(a) Show that the equations may be written in the conservation form:

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0, \qquad \frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2\right) = 0.$$

(b) Derive the energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}hu^2 + \frac{1}{2}gh^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}hu^3 + gh^2u\right) = 0.$$

(c) Consider a fixed interval (x_1, x_2) containing a hydraulic jump at $x_s(t)$, i.e., $(x_1 < x_s(t) < x_2)$. Show that the speed of the hydraulic jump $u_s = dx_s/dt$ has to satisfy the equations

$$u_s = \frac{\Delta(hu)}{\Delta h} \quad \text{and} \quad u_s = \frac{\Delta\left(hu^2 + \frac{1}{2}gh^2\right)}{\Delta(hu)},$$

where $\Delta(\varphi)$ denotes the discontinuity of the variable φ across the hydraulic jump.

(d) Consider a stationary hydraulic jump ($u_s = 0$), with u_1, h_1 on its left and u_2, h_2 on its right. Show that

$$u_1 = \pm\sqrt{\frac{gh_2}{2h_1}(h_1 + h_2)} \quad \text{and} \quad u_2 = \pm\sqrt{\frac{gh_1}{2h_2}(h_1 + h_2)}.$$

(e) From the energy equation, we see that the energy flux and rate of work done by the pressure force across unit width of the shallow water flow at location x is given by

$$E = \frac{1}{2}hu^3 + gh^2u.$$

Show that the net rate of energy input into a very slim interval containing the hydraulic jump is

$$E_1 - E_2 = \frac{gQ}{4h_1h_2}(h_2 - h_1)^3, \quad \text{where } Q = h_1u_1 = h_2u_2.$$

(f) Based on the equation for $(E_1 - E_2)$ above, explain why the flow across the stationary hydraulic jump must experience an increase in water depth.

4. (a) Consider the flow of an incompressible fluid governed by the Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0.$$

Derive the following equation for the local rate of change of the kinetic energy density:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}^2\right) + \nabla \cdot \left[\left(\frac{1}{2}\mathbf{u}^2 + \frac{p}{\rho}\right)\mathbf{u} + \nu \boldsymbol{\omega} \times \mathbf{u}\right] = -\nu \boldsymbol{\omega}^2$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

(b) Write a brief note on Kolmogorov's 1941 theory for fully developed, isotropic turbulence. State the basic assumptions and show how they lead to the following conclusions:

- The length scale η and the velocity scale v of the smallest eddies are related to the generation length scale ℓ and velocity scale u as

$$\eta \sim Re^{-3/4}\ell, \qquad v \sim Re^{-1/4}u.$$

- The inertial subrange energy spectrum $E(k)$ is expected to depend on the $[-5/3]$-power of the wavenumber k:

$$E(k) \sim \epsilon^{2/3}k^{-5/3},$$

where ϵ is the rate of energy cascade.

The following equations might be useful:

$$\mathbf{u} \cdot \nabla\varphi = \nabla(\varphi\mathbf{u}) - \varphi\nabla \cdot \mathbf{u};$$

$$\mathbf{u} \cdot \nabla\mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla\frac{1}{2}\mathbf{u}^2;$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{w}) = (\nabla \times \mathbf{u}) \cdot \mathbf{w} - (\nabla \times \mathbf{w}) \cdot \mathbf{u};$$

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2\mathbf{u}.$$

End of Paper