NATIONAL UNIVERSITY OF SINGAPORE

PC3238 – FLUID DYNAMICS

(Semester II: AY 2015–16)

Time Allowed: 2 Hours

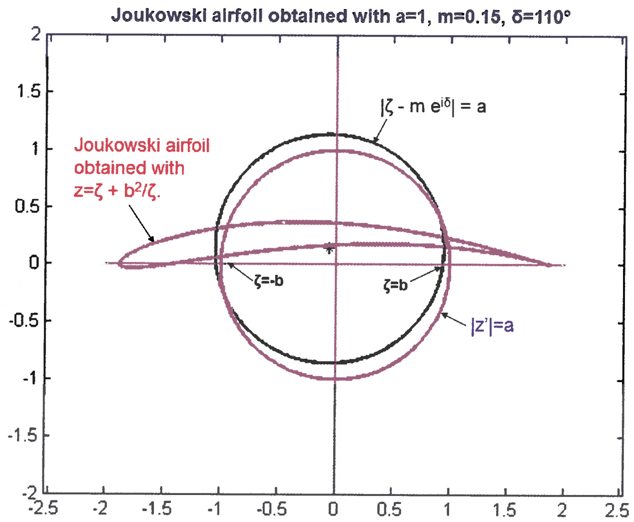
INSTRUCTIONS TO STUDENTS

1. Please write your student number only. **Do not write your name.**
2. This assessment paper contains FOUR (4) questions and comprises SIX (6) printed pages, inclusive of this cover page.
3. Students are required to answer any THREE (3) questions.
4. Students should write the answers for each question on a new page.
5. This is a CLOSED BOOK examination.
6. One Help Sheet (A4 size, both sides) is allowed for this examination.

1. The Joukowsky airfoil is obtained from the circle $|z'| = a$ in the z'-plane via the following two transformations

$$\begin{aligned} \zeta &= z' + c, \\ z &= \zeta + \frac{b^2}{\zeta}. \end{aligned}$$

The first transformation yields a circle in the ζ-plane centred at the point $\zeta = c$ (usually written in polar form, $c = me^{i\delta}$ with $m < a$ and $\pi/2 \leq \delta \leq \pi$). Let the shifted circle $|\zeta - c| = a$ intersect the positive real axis at $\zeta = b$. The value b then becomes the key parameter in the second transformation. An example of the transformations leading to an airfoil shape is shown in the figure below:



(a) Explain how the transformations create a sharp point at the trailing end of the airfoil shape ($z = 2b$).

(b) A flow with circulation $-\Gamma$ around the original circle in the z'-plane is given by the complex velocity potential:

$$w_0(z') = Uz' + U\frac{a^2}{z'} - \frac{\Gamma}{2\pi i} \ln z'.$$

The corresponding flow (with complex velocity potential $w(z)$) around the airfoil shape is obtained via the transformations:

$$w(z) = w_0\Big\{z'\big[\zeta(z)\big]\Big\}.$$

Show that a physically acceptable solution for the flow around the airfoil shape is obtained only when the circulation takes on a specific value equal to

$$\frac{\Gamma}{2\pi i} = Ua(\mathrm{e}^{-i\gamma} - \mathrm{e}^{i\gamma}) \quad \text{or} \quad \Gamma = 4\pi a U \sin\gamma,$$

where γ is defined by

$$a\mathrm{e}^{-i\gamma} = b - m\mathrm{e}^{i\delta}.$$

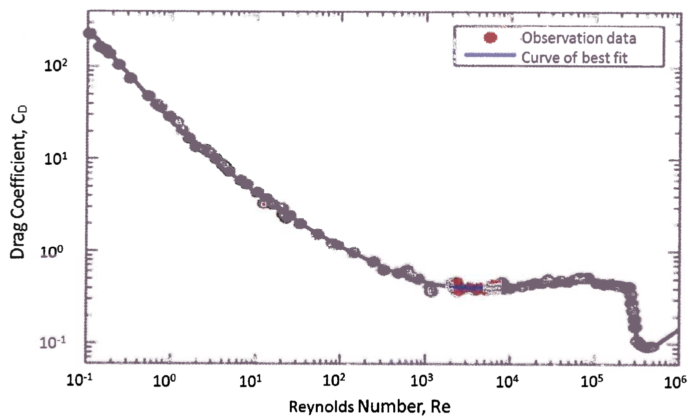
(c) Show that the flow velocity of the physically acceptable solution is given by

$$u - iv = \frac{U\left[(\zeta - c)(b - c) - a^2\right]\zeta^2}{(\zeta - c)^2(\zeta + b)(b - c)},$$

and at the trailing end of the airfoil shape ($z = 2b$ or $\zeta = b$), the velocity components are

$$u = \frac{Ub}{a}\cos\gamma\cos 2\gamma, \qquad v = -\frac{Ub}{a}\cos\gamma\sin 2\gamma.$$

2. The figure below is a log-log plot of the Drag Coefficient C_D versus the Reynolds Number Re for viscous flows around a sphere.



(a) Perform a dimensional analysis to derive the following expression for the drag D experienced by a sphere immersed in a steady flow of a viscous fluid:

$$\frac{D}{\rho U^2 a^2} = \Phi\left(\frac{\rho a U}{\mu}\right),$$

where $\Phi(x)$ is a general function of the argument x. We may assume that the relevant physical parameters are: the density ρ and the viscosity μ of the fluid, the flow speed U far upstream of the sphere, and the radius a of the sphere.

(b) The Stokes formula gives the drag on a sphere as $D = 6\pi\mu a U$. Show that the Stokes formula is a special case of the above general expression for the drag D.

(c) What should be the gradient of the log-log plot of C_D versus Re when the Stokes formula is valid? From the log-log plot above, state the range of values of the Reynolds Number Re where the Stokes formula is applicable.

(d) State the range of values of the Reynolds Number Re where the drag is approximately proportional to U^2.

(e) A sharp decrease in the Drag Coefficient is observed at Reynolds Number around 3×10^5. Briefly describe the physical phenomena that cause the sharp drop in drag.

3. The dispersion relation of deep water waves is given by

$$\omega^2 = gk + \frac{T}{\rho}k^3$$

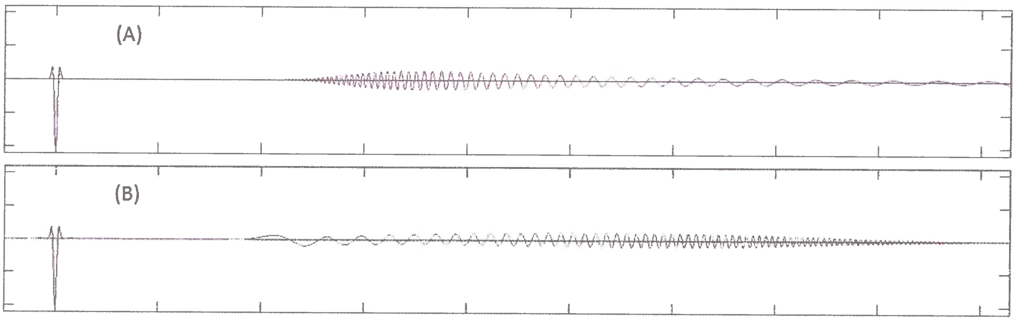
where g is the gravitational acceleration, T the surface tension of water, and ρ the density of water.

(a) Show that the phase velocity of deep water waves has a minimum value of about $0.23\,\mathrm{m\,s^{-1}}$ at wavelength of about 1.7 cm. (*Values of constants:* $g = 9.8\,m\,s^{-2}$, $T = 0.074\,N\,m^{-1}$, $\rho = 1{,}000\,kg\,m^{-3}$.)

(b) Show that for short wavelengths, $k \gg \sqrt{g\rho/T}$, we have capillary waves where the gravitational effect may be neglected. What are the phase velocity and group velocity of the capillary waves?

(c) For long wavelengths, $k \ll \sqrt{g\rho/T}$, we have gravity waves. What are the phase velocity and group velocity of gravity waves?

(d) Figures (A) and (B) below show the wavetrains that evolve from the initial disturbance on the left while they propagate to the right. Note that despite being shown side by side, the wavetrains are of very different length scales. State with explanation which wavetrain is gravity wave in nature and which is capillary wave in nature.



(e) A steady stream of speed U is disturbed by a stationary obstacle. Is there a minimum flow speed U for steady waves to be excited by the obstacle? If steady waves are excited, are the capillary waves found upstream or downstream? Why? Are the gravity waves found upstream or downstream? Why? If the flow speed U increases, how will it affect the wavelengths of the capillary waves and the gravity waves?

(f) If the length of the yacht in the figure below is 10 m, estimate the speed at which it is sailing:

4. (a) Consider the flow of an incompressible fluid governed by the Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0.$$

Derive the following equation for the local rate of change of the kinetic energy density:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}^2\right) + \nabla \cdot \left[\left(\frac{1}{2}\mathbf{u}^2 + \frac{p}{\rho}\right)\mathbf{u} + \nu\boldsymbol{\omega} \times \mathbf{u}\right] = -\nu\boldsymbol{\omega}^2$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

Briefly discuss the implications of the above equation on the viscous dissipation of kinetic energy of fluid flows.

(b) Write a brief note on Kolmogorov's 1941 theory for fully developed, isotropic turbulence. State the basic assumptions and show how they lead to the following conclusions:

- The length scale η and the velocity scale v of the smallest eddies are related to the generation length scale ℓ and velocity scale u as
$$\eta \sim Re^{-3/4}\ell, \qquad v \sim Re^{-1/4}u.$$
- The inertial subrange energy spectrum $E(k)$ is expected to depend on the $[-5/3]$-power of the wavenumber k:
$$E(k) \sim \epsilon^{2/3}k^{-5/3},$$
where ϵ is the rate of energy cascade.

The following equations might be useful:

$$\mathbf{u} \cdot \nabla\varphi = \nabla(\varphi\mathbf{u}) - \varphi\nabla \cdot \mathbf{u};$$

$$\mathbf{u} \cdot \nabla\mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla\frac{1}{2}\mathbf{u}^2;$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{w}) = (\nabla \times \mathbf{u}) \cdot \mathbf{w} - (\nabla \times \mathbf{w}) \cdot \mathbf{u};$$

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2\mathbf{u}.$$

– End of Paper –