NATIONAL UNIVERSITY OF SINGAPORE

PC3238 – FLUID DYNAMICS

(Semester II: AY 2017–18)

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. Please write your student number only. **Do not write your name.**
2. This assessment paper contains FOUR (4) questions and comprises SIX (6) printed pages, inclusive of this cover page.
3. Students are required to answer any THREE (3) questions.
4. Students should write the answers for each question on a new page.
5. This is a CLOSED BOOK examination.
6. One Help Sheet (A4 size, both sides) is allowed for this examination.

1. EITHER

(a) For 2-D irrotational flows of an incompressible and non-viscous fluid, the flow field $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$ may be written in terms of a complex velocity potential $w(z) = \phi(x, y) - i\psi(x, y)$, where $z = x + iy$, such that:

$$dw/dz = u - iv.$$

i. Show that the net pressure force components F_x and F_y on a 2-D object immersed in the flow may be evaluated by the following contour integrals around the object:

$$F_x - iF_y = \frac{i\rho}{2} \oint \left(\frac{dw}{dz}\right)^2 dz,$$

where ρ is the density of the fluid.

ii. Show that the transformation

$$z = \zeta + \frac{b^2}{\zeta^2}, \quad \text{where } b \text{ is a positive real number,}$$

transforms a circle of radius a $(a > b)$ in the ζ-plane to an ellipse in the z-plane.

iii. A flow around a circle of radius a in the ζ-plane is given by the complex velocity potential

$$w_0(\zeta) = U\zeta + U\frac{a^2}{\zeta} + \frac{\Gamma}{2\pi i} \ln \zeta.$$

Sketch the flow around the ellipse in the z-plane given by the complex velocity potential

$$w(z) = w_0[\zeta(z)], \quad \text{with} \quad z = \zeta + \frac{b^2}{\zeta^2}.$$

iv. Determine the lift and drag experienced by the ellipse.

<u>*Note that:*</u>

$$\oint f(z)\, dz = 2\pi i \sum \textit{Residues}.$$

OR

(b) Consider a single line vortex which initially occupies the z-axis. The vorticity depends only on r and t. The applicable vorticity equation in 2D in polar coordinates is then

$$\frac{\partial \zeta}{\partial t} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \zeta}{\partial r}\right).$$

i. The circulation of the vortex is defined as $\Gamma = \int_0^\infty \zeta(r,t)\, 2\pi r\, dr$. Show that the circulation remains constant with time:

$$\frac{d\Gamma}{dt} = \int_0^\infty \frac{\partial \zeta}{\partial t} 2\pi r\, dr = 0.$$

ii. Taking note that the physical parameters involved in this problem are ζ, ν, r and t, perform a dimensional analysis to propose a solution in the special form:

$$\zeta = \frac{1}{t} f(\eta), \quad \text{where } f \text{ is a general function, and } \eta = \frac{r}{\sqrt{\nu t}}.$$

iii. Substitute the above into the vorticity equation to derive the following equation for the function f:

$$2\frac{d}{d\eta}\left(\eta\frac{df}{d\eta}\right) + \left(2\eta f + \eta^2\frac{df}{d\eta}\right) = 0.$$

iv. Integrate the above equation once to give

$$2\eta\frac{df}{d\eta} + \eta^2 f = A,$$

where A is the integration constant. Explain why we should set $A = 0$.

v. Integrate the equation of the last step to obtain

$$f(\eta) = B\exp\left(-\frac{\eta^2}{4}\right).$$

vi. Express the solution for the vorticity distribution in terms of the circulation:

$$\zeta(r,t) = \frac{\Gamma}{4\pi\nu t}\exp\left(-\frac{r^2}{4\nu t}\right).$$

2. The equations for small-amplitude gravity waves in calm water of depth H are:

$$\begin{aligned} \nabla^2\phi &= 0 && \text{for all } x,\ -H < z < 0; \\ \frac{\partial\phi}{\partial z} &= 0 && \text{at } z = -H; \\ \frac{\partial\phi}{\partial z} &= \frac{\partial\eta}{\partial t} && \text{at } z = 0; \\ \frac{\partial\phi}{\partial t} + g\eta &= 0 && \text{at } z = 0; \end{aligned}$$

where $\phi(x, z, t)$ is the velocity potential, $\eta(x, t)$ is the surface perturbation, ρ the density of water, and g the gravitational acceleration.

(a) Obtain the following solution for the surface perturbation and the velocity potential:

$$\begin{aligned} \eta(x,t) &= \eta_0 \cos(\omega t - kx), \\ \phi(x,z,t) &= -\frac{\eta_0\omega}{k\sinh(kH)} \cosh\left[k(z+H)\right] \sin(\omega t - kx). \end{aligned}$$

(b) Derive the water wave dispersion relation

$$\omega^2 = gk\tanh(kH).$$

(c) Show that in shallow water, gravity waves are non-dispersive and have a phase velocity of $c = \sqrt{gH}$.

(d) Show that in deep water, gravity waves are dispersive, and have group velocity equal to half of the corresponding phase velocity.

(e) Derive the following expression for the velocity components:

$$\begin{aligned} u(x,z,t) &= \frac{\eta_0\omega}{\sinh(kH)} \cosh\left[k(z+H)\right] \cos(\omega t - kx), \\ w(x,z,t) &= -\frac{\eta_0\omega}{\sinh(kH)} \sinh\left[k(z+H)\right] \sin(\omega t - kx). \end{aligned}$$

(f) Show that the average kinetic energy in a column of water with unit area cross section is

$$\int_{-H}^{0} \frac{1}{2}\rho\left\langle u^2 + w^2\right\rangle dz = \frac{1}{4}\rho g \eta_0^2.$$

3. Two-dimensional flow of an incompressible, non-viscous fluid may be studied with the vorticity equation expressed in terms of a stream function $\tilde{\psi}$:

$$\left[\frac{\partial}{\partial t}+\frac{\partial\tilde{\psi}}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\tilde{\psi}}{\partial y}\frac{\partial}{\partial x}\right]\nabla^2\tilde{\psi}=0,$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, and the flow velocity components are given by $u=-\frac{\partial\tilde{\psi}}{\partial y}$ and $v=\frac{\partial\tilde{\psi}}{\partial x}$.

(a) Show that a parallel flow $U(y)$ between two impenetrable boundaries at $y=-H$ and $y=H$ satisfies the equation.

(b) Derive the following linearized equation for a small perturbation $\psi(x,y,t)$ on the parallel flow $U(y)$:

$$\left[\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right]\nabla^2\psi-U''\frac{\partial\psi}{\partial x}=0, \quad \text{where } U''=\frac{d^2U}{dy^2}.$$

State the boundary conditions to be satisfied.

(c) Let $\psi(x,y,t)=\Phi(y)\mathrm{e}^{ik(x-ct)}$. Show that the equation for $\Phi(y)$ is

$$\left[(U-c)(D^2-k^2)-U''\right]\Phi(y)=0, \quad \text{where } D=\frac{d}{dy}.$$

State the boundary conditions to be satisfied.

(d) If the parallel flow $U(y)$ is unstable, then $c=c_r+ic_i$, so that $\mathrm{e}^{ik(x-ct)}=\mathrm{e}^{ik(x-c_rt)}\mathrm{e}^{c_it}$. We may then write the above equation as

$$L(\Phi)=(D^2-k^2)\Phi-\frac{U''}{U-c}\Phi=0.$$

Taking the complex conjugate, we have

$$L^*(\Phi^*)=(D^2-k^2)\Phi^*-\frac{U''}{U-c^*}\Phi^*=0.$$

Show that

$$\Phi^*L(\Phi)-\Phi L^*(\Phi^*)=D(\Phi^*D\Phi-\Phi D\Phi^*)-2ic_i\frac{U''|\Phi|^2}{|U-c|^2}.$$

(e) Show that a necessary condition for the flow to be unstable is

$$\int_{-H}^{H}\frac{U''|\Phi|^2}{|U-c|^2}\,dy=0.$$

4. (a) For an incompressible Newtonian fluid, the stress tensor is given by

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu S_{ij}, \quad \text{where} \quad S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Derive the kinetic energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_i u_i\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho u_i u_i u_j + p u_j - 2\mu\, u_i S_{ij}\right) = -2\mu\, S_{ij} S_{ij}.$$

Integrating over a volume V which includes all locations where the flow velocity does not vanish, show that the rate of change of the total kinetic energy of the flow is given by

$$\frac{d}{dt}\int_V \frac{1}{2}\rho\, u_j u_j\, dV = -\int_V 2\mu\, S_{ij} S_{ij}\, dV.$$

(b) Write a brief note on Kolmogorov's 1941 theory for fully developed, isotropic turbulence. State the basic assumptions and show how they lead to the following conclusions:

- The length scale η and the velocity scale v of the smallest eddies are related to the generation length scale ℓ and velocity scale u as

$$\eta \sim Re^{-3/4}\ell, \qquad v \sim Re^{-1/4}u.$$

- The inertial subrange energy spectrum $E(k)$ is expected to depend on the $[-5/3]$-power of the wavenumber k:

$$E(k) \sim \epsilon^{2/3} k^{-5/3},$$

where ϵ is the rate of energy cascade.

- Based on the above conclusions, discuss the feasibility of Direct Numerical Simulation of turbulent flow of high Reynolds number.

– End of Paper –

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