**NATIONAL UNIVERSITY OF SINGAPORE**

PC3238 – FLUID DYNAMICS

(Semester II: AY 2018–19)

Time Allowed: 2 Hours

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## INSTRUCTIONS TO STUDENTS

1. Please write your student number only. **Do not write your name.**
2. This assessment paper contains FOUR (4) questions and comprises SEVEN (7) printed pages, inclusive of this cover page.
3. Students are required to answer any THREE (3) questions.
4. Students should write the answers for each question on a new page.
5. This is a CLOSED BOOK examination.
6. One Help Sheet (A4 size, both sides) is allowed for this examination.

1. EITHER

(a) For an incompressible, non-viscous fluid of constant and homogeneous density $\rho_0$, the equations governing the flows (with vertically upward $z$-axis) are

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} &= -\nabla\left(\frac{p}{\rho_0}\right) - \nabla(gz), \\ \nabla \cdot \mathbf{v} &= 0.\end{aligned}$$

Show that the momentum equation may be written as

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{v}^2 + \frac{p}{\rho_0} + gz\right) = \mathbf{v} \times \boldsymbol{\omega},$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is the vorticity of the flow field.

(b) Derive the following versions of Bernoulli's equation, with clear statements of the conditions for each of them to be valid:

$$\begin{aligned}\frac{1}{2}\mathbf{v}^2 + \frac{p}{\rho_0} + gz &= \text{constant}, \\ \mathbf{v} \cdot \nabla\left(\frac{1}{2}\mathbf{v}^2 + \frac{p}{\rho_0} + gz\right) &= 0, \\ \frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho_0} + gz &= 0\end{aligned}$$

where $\phi$ is velocity potential which gives $\mathbf{v} = \nabla\phi$.

(c) When the viscous effect cannot be neglected, the momentum equation becomes

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\left(\frac{p}{\rho_0}\right) - \nabla(gz) + \nu\nabla^2\mathbf{v}.$$

Show that for a steaty-state, irrotational flow of this incompressible fluid, the following equation is still valid

$$\frac{1}{2}\mathbf{v}^2 + \frac{p}{\rho_0} + gz = \text{constant}.$$

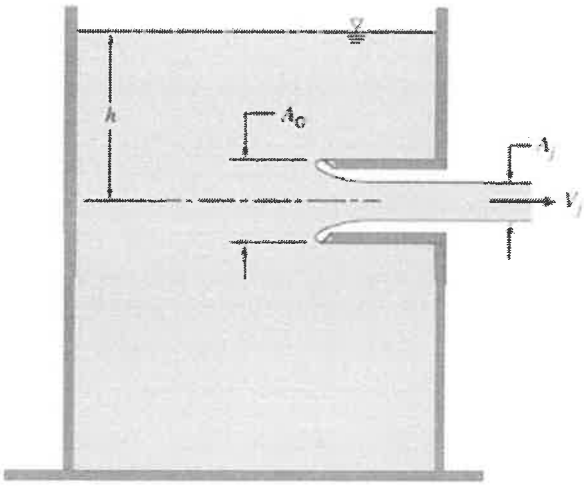
(d) Does the symbol $p$ in the equations in Part (c) represent an isotropic pressure in the fluid? If not, what does it represent?
*Hint:* $\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

OR

(a) Water at atmospheric pressure is rushing into a vacuum through an exit. Estimate the flow speed at the exit. (Density of water = 1,000 kg m$^{-3}$, atmospheric pressure = 101,325 Pa.)

(b) What will be the flow speed if it is air instead of water? The air flow may be assumed to be adiabatic. (Density of air = 1.2 kg m$^{-3}$, $\gamma = c_p/c_v = 1.4$; atmospheric pressure = 101,325 Pa.)

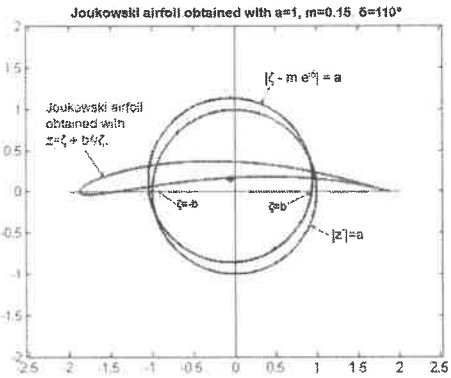
(c) The figure below shows a vessel filled with water. Water is discharging from the vessel through a Borda mouthpiece (a circular orifice with a cylindrical tube projecting inward) at a depth $h$ below the water surface in the vessel. The cross-sectional area of the Borda mouthpiece is $A_0$. Determine the discharge rate of water from the vessel. Please take care to explain the basic principles involved in the determination of the discharge rate.



2. The Joukowsky airfoil is obtained from the circle $|z'| = a$ in the $z'$-plane via the following two transformations

$$\zeta = z' + c, \qquad \text{and} \qquad z = \zeta + \frac{b^2}{\zeta}.$$

The first transformation yields a circle in the $\zeta$-plane centred at the point $\zeta = c$ (usually written in polar form, $c = me^{i\delta}$ with $m < a$ and $\pi/2 \le \delta \le \pi$). Let the shifted circle $|\zeta - c| = a$ intersect the positive real axis at $\zeta = b$. The value $b$ then becomes the key parameter in the second transformation. An example of the transformations leading to an airfoil shape is shown in the figure below:




(a) Explain how the transformations create a sharp point at the trailing end of the airfoil shape ($z = 2b$).

(b) A flow with circulation $-\Gamma$ around the original circle in the $z'$-plane is given by the complex velocity potential:

$$w_0(z') = Uz' + U\frac{a^2}{z'} - \frac{\Gamma}{2\pi i}\ln z'.$$

The corresponding flow (with complex velocity potential $w(z)$) around the airfoil shape is obtained via the transformations:

$$w(z) = w_0\Big\{z'\big[\zeta(z)\big]\Big\}.$$

Show that a physically acceptable solution for the flow around the airfoil shape is obtained only when the circulation takes on a specific value equal to

$$\frac{\Gamma}{2\pi i} = Ua(\mathrm{e}^{-i\gamma} - \mathrm{e}^{i\gamma}) \quad \text{or} \quad \Gamma = 4\pi aU\sin\gamma,$$

where $\gamma$ is defined by $a\mathrm{e}^{-i\gamma} = b - me^{i\delta}$.

3. EITHER

The equations for small-amplitude gravity waves in calm water of depth $H$ are:

$$\begin{aligned} \nabla^2\phi &= 0 && \text{for all } x,\ -H < z < 0; \\ \frac{\partial\phi}{\partial z} &= 0 && \text{at } z = -H; \\ \frac{\partial\phi}{\partial z} &= \frac{\partial\eta}{\partial t} && \text{at } z = 0; \\ \frac{\partial\phi}{\partial t} + g\eta &= 0 && \text{at } z = 0; \end{aligned}$$

where $\phi$ is the velocity potential, $\eta$ is the surface perturbation, $\rho$ the density of water, and $g$ the gravitational acceleration.

(a) Obtain the following solution for the surface perturbation and the velocity potential:

$$\begin{aligned} \eta(x,t) &= \eta_0 \cos(\omega t - kx), \\ \phi(x,z,t) &= -\frac{\eta_0\omega}{k\sinh(kH)}\cosh\left[k(z+H)\right]\sin(\omega t - kx). \end{aligned}$$

(b) Derive the following gravity wave dispersion relation and discuss the dispersive properties of shallow and deep water gravity waves:

$$\omega^2 = gk\tanh(kH).$$

(c) Derive the following expression for the velocity components for a *deep water* gravity wave:

$$\begin{aligned} u(x,z,t) &= \eta_0\,\omega\,\mathrm{e}^{kz}\cos(\omega t - kx), \\ w(x,z,t) &= -\eta_0\,\omega\,\mathrm{e}^{kz}\sin(\omega t - kx). \end{aligned}$$

(d) It is known that, to the first approximation, a small parcel of water makes circular motion about a mean position $(x_c, z_c)$ on the passage of a gravity wave. Let the coordinates of the small parcel of water be $x_p = x_c + x_d$, $z_p = z_c + z_d$. Show that the circular motion is given by the

$$x_d = \eta_0\mathrm{e}^{kz_c}\sin(\omega t - kx_c), \qquad z_d = \eta_0\mathrm{e}^{kz_c}\cos(\omega t - kx_c).$$

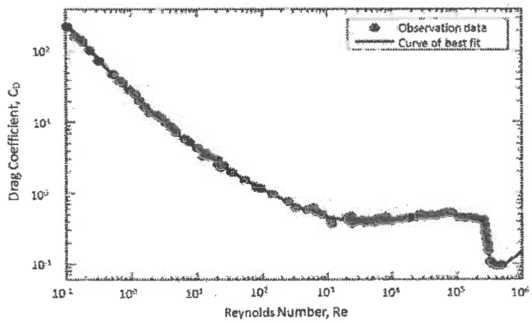
(e) Show that the gravity wave induces a drift current of magnitude

$$u_{drift} = \eta_0^2\,\omega\,k\,\mathrm{e}^{2kz}$$

by taking the next higher order terms in the equations for $x_d$ and $z_d$ into consideration.

OR

The figure below is a log-log plot of Drag Coefficient $C_D = D/(\rho U^2 a^2)$ versus Reynolds Number $Re = Ua\rho/\mu$ for viscous flow around a sphere.




(a) Consider a viscous fluid of density $\rho$ and viscosity $\mu$ flowing around a sphere of radius $a$. The flow speed far from the sphere is $U$. Perform a dimensional analysis to show that the drag $D$ experienced by the sphere is expected to depend on the various parameters in the following manner:

$$\frac{D}{\rho U^2 a^2} = \Phi\left(\frac{\rho a U}{\mu}\right),$$

where $\Phi(x)$ is a general function of the argument $x$.

(b) The Stokes formula gives the drag on a sphere as $D = 6\pi\mu a U$. What is the function $\Phi(x)$ in Part (i) when the Stokes formula is valid?

(c) What should be the gradient of the plot when the Stokes formula is valid? State the range of values of the Reynolds Number where the Stokes formula is applicable.

(d) State the range of values of the Reynolds Number where the drag is approximately proportional to $U^2$.

(e) Given the following values of the parameters $\rho = 1,000\,\mathrm{kg\,m^{-3}}$, $\mu = 0.001\,\mathrm{kg\,m^{-1}s^{-1}}$, $a = 0.02\,\mathrm{m}$, estimate the drag forces for $U = 0.5\,\mathrm{m\,s^{-1}}, 10\,\mathrm{m\,s^{-1}}$, and $20\,\mathrm{m\,s^{-1}}$.

(f) A sharp decrease in the Drag Coefficient is observed at Reynolds Number around $2.5\times10^5$. Briefly describe the physical phenomena that cause the sharp drop in drag.

4. (a) For an incompressible Newtonian fluid, the stress tensor is given by

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu S_{ij}, \quad \text{where} \quad S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Derive the kinetic energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_i u_i\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho u_i u_i u_j + p u_j - 2\mu\, u_i S_{ij}\right) = -2\mu\, S_{ij} S_{ij}.$$

Integrating over a volume $V$ which includes all locations where the flow velocity does not vanish, show that the rate of change of the total kinetic energy of the flow is given by

$$\frac{d}{dt}\int_V \frac{1}{2}\rho\, u_j u_j\, dV = -\int_V 2\mu\, S_{ij} S_{ij}\, dV.$$

(b) Write a brief note on Kolmogorov's 1941 theory for fully developed, isotropic turbulence. State the basic assumptions and show how they lead to the following conclusions:

- The length scale $\eta$ and the velocity scale $v$ of the smallest eddies are related to the generation length scale $\ell$ and velocity scale $u$ as

$$\eta \sim Re^{-3/4}\ell, \qquad v \sim Re^{-1/4}u.$$

- The inertial subrange energy spectrum $E(k)$ is expected to depend on the $[-5/3]$-power of the wavenumber $k$:

$$E(k) \sim \epsilon^{2/3} k^{-5/3},$$

  where $\epsilon$ is the rate of energy cascade.
- Based on the above conclusions, discuss the feasibility of Direct Numerical Simulation of turbulent flow of high Reynolds number.

– End of Paper –

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