

PC3274

NATIONAL UNIVERSITY OF SINGAPORE

PC3274 – Mathematical Methods 2

(Lecturer: B.-G. Englert)

(Semester I: AY2006/07)

Exam, 27 November 2006

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR** questions and comprises **THREE** printed pages.
2. Answer **ALL FOUR** questions for a total of 100 marks.
3. Show all your work in the answer book.
4. For each question, **clearly** indicate what constitutes your final answer.
5. A single A4-size cheat sheet may be consulted during the exam, **but no other written or printed material**.
6. The use of **electronic equipment** of any kind **is not permitted**.

1. Calculus of Variations (15 marks)

For given $a > 0$, what is the smallest value that you can get for

$$\int_{-a}^a dx \left[\frac{d}{dx} y(x) \right]^2$$

if the permissible $y(x)$ are restricted by

$$y(\pm a) = 0 \quad \text{and} \quad \int_{-a}^a dx |x| y(x) = a^3?$$

2. Group Theory (35=15+15+5 marks)

The elements $g = (a, b)$ of the set G are ordered pairs of complex numbers, subject to $|a|^2 - |b|^2 = 1$, one element of G for each a, b pair. The composition of two elements of G , written as a product, is defined by

$$g_1 g_2 = (a_1 a_2 + b_1^* b_2, b_1 a_2 + a_1^* b_2) \quad \text{for} \quad g_1 = (a_1, b_1), g_2 = (a_2, b_2).$$

- (a) Show that, with this composition rule, G is a group. In particular, state the a, b pairs for the neutral element e and the inverse g^{-1} of g .
- (b) Consider the following four restrictions on the values of a and b :

- (i) $\text{Im}(a) = \text{Im}(b)$;
- (ii) $\text{Im}(a) = -\text{Im}(b)$;
- (iii) $\text{Im}(b) = 0$;
- (iv) $b = 0$.

Which of them define subgroups of G ?

- (c) Of those restrictions that define subgroups, which ones define abelian subgroups?

3. Laplace Transform (15 marks)

For $T > 0$, find the Laplace transform $F(s)$ of

$$f(t) = \begin{cases} 1 & \text{if } \sin(2\pi t/T) > 0, \\ -1 & \text{if } \sin(2\pi t/T) < 0. \end{cases}$$

4. Contour Integration and Laplace Transformation (35=10+15+10 marks)

The functions $b_n(t)$ with $n = 0, \pm 1, \pm 2, \dots$ are defined by their generating function in accordance with

$$e^{\frac{1}{2}t(z - 1/z)} = \sum_{n=-\infty}^{\infty} z^n b_n(t) = \sum_{n=-\infty}^{\infty} (-z)^{-n} b_n(t).$$

(a) Show that

$$(-1)^n b_n(t) = b_{-n}(t) = \int_{\mathcal{C}} \frac{dz}{2\pi iz} z^n e^{\frac{1}{2}t(z - 1/z)},$$

where the contour \mathcal{C} circles $z = 0$ once counter-clockwise along the unit circle.

(b) Use this to calculate the Laplace transforms $B_n(s)$ of all $b_n(t)$.

(c) Evaluate the integrals $\int_0^{\infty} dt b_0(t)$ and $\int_0^{\infty} dt \frac{b_1(t)}{t}$.

Hint: Do not worry about interchanging the order of summations and integrations.

End of Paper