Question 1

(a) Use the method of Laplace transform to find a particular integral of the equation

$$y'' + 2ay' + a^2 y = f(t).$$

(b) Show that the non-linear differential equation

$$y''(t) + y^2(t) = t\sin t,$$

given that $y(0) = 1$ and $y'(0) = -1$, can be written as the integral equation

$$y(t) + \int_0^t (t-u)y^2(u)\,du = 3 - t - 2\cos t - t\sin t.$$

a) $y'' + 2ay' + a^2 y = f(t)$

$$s^2\tilde{y} - sy_0 - y_0' + 2as\tilde{y} - y_0 + a^2\tilde{y} = \tilde{f}$$

$$(s^2 + 2as + a^2)\tilde{y} = \tilde{f} + (s+1)y_0 + y_0'$$

$$\tilde{y} = \frac{\tilde{f}}{(s+a)^2} + \frac{s+1}{(s+a)^2}y_0 + \frac{y_0'}{(s+a)^2}$$

$$\tilde{y} = \frac{\tilde{f}}{(s+a)^2} + \frac{(1-a)y_0 + y_0'}{(s+a)^2} + \frac{y_0}{s+a}$$

$$y = \int_0^t (t-u)f(u)e^{-a(t-u)}\,du + t([(1-a)y_0 + y_0']e^{-at} + y_0 e^{-at}$$

b) $y'' + y^2 = t\sin t$

$$s^2\tilde{y} - sy(0) - y'(0) + \widetilde{y^2} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$s^2\tilde{y} - s + 1 + \widetilde{y^2} = \frac{2s}{(s^2+1)^2}$$

$$\tilde{y} + \frac{\widetilde{y^2}}{s^2} = \frac{2}{s(s^2+1)^2} + \frac{1}{s} - \frac{1}{s^2}$$

$$= \frac{2}{s} - \frac{2s}{s^2+1} - \frac{2s}{(s^2+1)^2} + \frac{1}{s} - \frac{1}{s^2}$$

$$= \frac{3}{s} - \frac{1}{s^2} - \frac{2s}{s^2+1} - \frac{2s}{(s^2+1)^2}$$

$$\therefore y + \int_0^t (t-u)y^2(u)\,du = 3 - t - 2\cos t - t\sin t$$

Question 2

(a) Show that if $\mathcal{G}$ is a finite group of order g, and $\mathcal{H}$ is a subgroup of $\mathcal{G}$ and of order h, then g is a multiple of h.

(b) Show that the following set of six functions,

$$f_1(x) = x, \qquad f_2(x) = \frac{1}{1-x}, \qquad f_3(x) = \frac{x-1}{x},$$
$$f_4(x) = \frac{1}{x}, \qquad f_5(x) = 1-x, \qquad f_6(x) = \frac{x}{x-1}$$

with the law of combination as $f_i(x) \bullet f_j(x) = f_i[f_j(x)]$ forms a non-Abelian group. Determine the order of each element in the group.

(c) Show that the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

can be written as

$$\frac{\partial F}{\partial x} + \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'} - F\right) = 0.$$

where $F = F(y, y', x)$.

a) If $\mathcal{G}$ has order g, then the order of the subgroups must divide g. Given X & Y belong to $\mathcal{G}$ and $X \sim Y$, then $X, Y, X^{-1}Y$ belong to $\mathcal{H}$.

We let $Y = X\mathcal{H}_i$ for some element $\mathcal{H}_i$. Each coset must have distinct elements such that $\mathcal{H}$ must have h elements.

Since each member must only be in one set, then g is a multiple of h. [proven]

b) The table below shows that the operation is closed. Functions are always associative, the identity exists, and it is $f_1(x)$; every element has an inverse, and the table is not symmetric. ∴ the set of functions form a non-Abelian group.

	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$
$f_1(x)$	x	$\frac{1}{1-x}$	$\frac{x-1}{x}$	$\frac{1}{x}$	$1-x$	$\frac{x}{x-1}$
$f_2(x)$	$\frac{1}{1-x}$	$\frac{x-1}{x}$	x	$\frac{x}{x-1}$	$\frac{1}{x}$	$1-x$
$f_3(x)$	$\frac{x-1}{x}$	x	$\frac{1}{1-x}$	$1-x$	$\frac{x}{x-1}$	$\frac{1}{x}$
$f_4(x)$	$\frac{1}{x}$	$1-x$	$\frac{x}{x-1}$	x	$\frac{1}{1-x}$	$\frac{x-1}{x}$
$f_5(x)$	$1-x$	$\frac{x}{x-1}$	$\frac{1}{x}$	$\frac{x-1}{x}$	x	$\frac{1}{1-x}$
$f_6(x)$	$\frac{x}{x-1}$	$\frac{1}{x}$	$1-x$	$\frac{1}{1-x}$	$\frac{x-1}{x}$	x

Elements of

Order 1: $f_1(x)$

Order 2: $f_2(x), f_3(x)$

Order 3: $f_4(x), f_5(x), f_6(x)$

c) We know that

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y''$$

$$\frac{\partial F}{\partial y} = \frac{1}{y'}\frac{dF}{dx} - \frac{y''}{y'}\frac{\partial F}{\partial y'} - \frac{1}{y'}\frac{\partial F}{\partial x}$$

then

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\frac{1}{y'}\frac{dF}{dx} - \frac{y''}{y'}\frac{\partial F}{\partial y'} - \frac{1}{y'}\frac{\partial F}{\partial x} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\frac{dF}{dx} - \left[y''\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] - \frac{\partial F}{\partial x} = 0$$

We also find that

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y''\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

substituting back, we have

$$\frac{dF}{dx} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0$$

$$\frac{\partial F}{\partial x} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'} - F\right) = 0 \quad [shown]$$

Question 3

Find the extremal of the following functional

$$I = \int_0^1 (y'^2 + z'^2 - 4xz' - 4z)\,dx$$

subjected to the constraint

$$\int_0^1 (y'^2 - xy' - z'^2)\,dx = 2,$$

given that $y(0) = 0,\ z(0) = 0,\ y(1) = 1$ and $z(1) = 1$. Calculate the corresponding value of the integral I.

$F = y'^2 + z'^2 - 4xz' - 4z + \lambda(y'^2 - xy' - z'^2)$

$\frac{d}{dx}\frac{\partial F}{\partial y'} = \frac{\partial F}{\partial y}$ $\frac{d}{dx}(2y' + 2\lambda y' - \lambda x) = 0$ $2y' + 2\lambda y' - \lambda x = k$ $y' = \frac{1}{2+2\lambda}(k + \lambda x)$ $y = \frac{1}{2(1+\lambda)}\left(kx + \frac{\lambda x^2}{2} + c\right)$ using the boundary conditions, $y(0) = 0, y(1) = 1,$ $0 = \frac{c}{2(1+\lambda)} \Rightarrow c = 0$ $1 = \frac{1}{2(1+\lambda)}\left(k + \frac{\lambda}{2}\right)$ $4 + 4\lambda = 2k + \lambda$ $k = \frac{4+3\lambda}{2}$	$\frac{d}{dx}\frac{\partial F}{\partial z'} = \frac{\partial F}{\partial z}$ $\frac{d}{dx}(2z' - 4x - 2\lambda z') = -4$ $(2 - 2\lambda)z' - 4x = -4x + d$ $z' = \frac{d}{2-2\lambda}$ $z = \frac{d}{2-2\lambda}x + e$ using the boundary conditions, $z(0) = 0, z(1) = 1,$ $e = 0, \quad d = 2 - 2\lambda$ $\therefore z = x$

$\therefore y=\frac{1}{2(1+\lambda)}\left(\frac{4+3\lambda}{2}x+\frac{\lambda}{2}x^2\right)$ $=\frac{1}{4}\frac{\lambda}{1+\lambda}x^2+\frac{1}{4}\left(\frac{4+3\lambda}{1+\lambda}\right)x$ $=\frac{1}{4}\frac{1-1+\lambda}{1+\lambda}x^2+\frac{1}{4}\left(\frac{1+3+3\lambda}{1+\lambda}\right)x$ $=\frac{1}{4}\left(1-\frac{1}{1+\lambda}\right)x^2+\frac{1}{4}\left(3+\frac{1}{1+\lambda}\right)x$	

substituting $A=\frac{1}{1+\lambda}$, we have

$$y=\frac{1}{4}(1-A)x^2+\frac{1}{4}(3+A)x,\qquad z=x$$

$$y'=\frac{1}{2}(1-A)x+\frac{1}{4}(3+A),\qquad z'=1$$

$$\int_0^1(y'^2-xy'-z'^2)\,dx=2$$

$$\int_0^1\left(\left[\frac{1}{2}(1-A)x+\frac{1}{4}(3+A)\right]^2-x\left[\frac{1}{2}(1-A)x+\frac{1}{4}(3+A)\right]-1\right)dx=2$$

$$\int_0^1\left(\frac{1}{4}(1-A)^2x^2+\frac{1}{4}(1-A)(3+A)x+\frac{1}{16}(3+A)^2-\frac{1}{2}(1-A)x^2-\frac{1}{4}(3+A)x-1\right)dx=2$$

$$\int_0^1(4(A^2-1)x^2-4A(3+A)x+(3+A)^2-16)\,dx=32$$

$$\left[\frac{4}{3}(A^2-1)x^3-2A(3+A)x^2+[(3+A)^2-16]x\right]_0^1=32$$

$$4(A^2-1)-6A(3+A)+3(3+A)^2-48=96$$

$$4A^2-4-18A-6A^2+27+18A+3A^2-48-96=0$$

$$A^2-121=0,\qquad A=\pm 11,\qquad \lambda=\frac{1}{\pm 11}-1=-\frac{10}{11},-\frac{12}{11}$$

and we have

$$y=-\frac{5}{2}x^2+\frac{7}{2}x\quad or\qquad y=3x^2-2x$$

$$y'=-5x+\frac{7}{2}\quad or\quad y'=6x-2$$

For $y=-\frac{5}{2}x^2+\frac{7}{2}x$,

$$I=\int_0^1(y'^2+z'^2-4xz'-4z)\,dx$$

$$=\int_0^1\left(-5x+\frac{7}{2}\right)^2+1-8x\,dx$$

$$= \left[-\frac{1}{15}\left(-5x+\frac{7}{2}\right)^3 + x - 4x^2\right]_0^1$$

$$= -\frac{111}{40} + \frac{343}{120} = \frac{1}{12}$$

For $y = 3x^2 - 2x$,

$$I = \int_0^1 (y'^2 + z'^2 - 4xz' - 4z)\,dx$$

$$= \int_0^1 (6x-2)^2 + 1 - 8x\,dx$$

$$= \left[\frac{1}{18}(6x-2)^3 + x - 4x^2\right]_0^1$$

$$= -\frac{59}{9} - \frac{4}{9} = -7$$

Question 4

(a)

i. Represent the following function as an exponential Fourier transform

$$f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & otherwise \end{cases}$$

ii. Show that your results can be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos kx + \cos k(x-\pi)}{1-k^2}\,dk$$

(b) Use the method of tensor to establish the following vector identity,

$$grad\,\frac{1}{2}(\vec{u}\cdot\vec{u}) = \vec{u}\times curl\,\vec{u} + (\vec{u}\cdot grad)\vec{u}.$$

a)

i.

$$\tilde{f}(k) = \sqrt{\frac{1}{2\pi}}\int_0^\pi \sin x\, e^{-ikx}\,dx$$

$$= \sqrt{\frac{1}{2\pi}}\left[\frac{e^{-ikx}}{k^2-1}(ik\sin x + \cos x)\right]_0^{\pi}$$

$$= \sqrt{\frac{1}{2\pi}}\left(\frac{e^{-ik\pi}+1}{1-k^2}\right)$$

ii.

$$f(x) = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(k)e^{ikx}dk$$

$$= \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{1}{2\pi}}\left(\frac{e^{-ik\pi}+1}{1-k^2}\right)e^{ikx}dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ik(x-\pi)}+e^{ikx}}{1-k^2}dk$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\frac{\cos kx + \cos k(x-\pi)}{1-k^2}dk$$

b)

$$\vec{u}\times(\nabla\times\vec{u}) + (\vec{u}\cdot\nabla)\vec{u} = \epsilon_{ijm}\epsilon_{mnl}u_j\frac{\partial u_l}{\partial x_n} + u_j\frac{\partial u_i}{\partial x_j}$$

$$= \left(\delta_{in}\delta_{jl} - \delta_{il}\delta_{jn}\right)u_j\frac{\partial u_l}{\partial x_n} + u_j\frac{\partial u_i}{\partial x_j}$$

$$= u_j\frac{\partial u_j}{\partial x_i} - u_j\frac{\partial u_i}{\partial x_j} + u_j\frac{\partial u_i}{\partial x_j}$$

$$= u_j\frac{\partial u_j}{\partial x_i}$$

$$= \frac{1}{2}\left(u_j\frac{\partial u_j}{\partial x_i} + \frac{\partial u_j}{\partial x_i}u_j\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial x_i}(u_ju_j)$$

$$= \frac{1}{2}\nabla(\vec{u}\cdot\vec{u})$$

Solutions provided by: **A/Prof Paul Lim** (Q1, Q4) and **John Soo** (Q2, Q3)