NATIONAL UNIVERSITY OF SINGAPORE

PC3274 – MATHEMATICAL METHODS IN PHYSICS 2

(Semester I: AY 2008-09)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains FOUR questions and comprises FOUR printed pages.
2. Answer any THREE questions.
3. All questions carry equal marks.
4. Answers to the questions are to be written in the answer books.
5. This is a CLOSED BOOK examination.
6. A table of constants is provided.
7. The last page contains a list of formulae.

1. (a) Use the method of Laplace transform to find a particular integral of the equation

$$y'' + 2ay' + a^2 y = f(t).$$

(b) Show that the non-linear differential equation

$$y''(t) + y^2(t) = t \sin t,$$

given that $y(0) = 1$ and $y'(0) = -1$, can be written as the integral equation

$$y(t) + \int_0^t (t-u) y^2(u)\, du = 3 - t - 2\cos t - t \sin t.$$

2. (a) Show that if $\mathscr{G}$ is a finite group of order g, and $\mathscr{H}$ is a subgroup of $\mathscr{G}$ and of order h, then g is a multiple of h.

(b) Show that the following set of six functions,
$f_1(x) = x$, $f_2(x) = 1/(1-x)$,
$f_3(x) = (x-1)/x$, $f_4(x) = 1/x$,
$f_5(x) = 1 - x$, $f_6(x) = x/(x-1)$,
with the law of combination as $f_i(x) \bullet f_j(x) = f_i(f_j(x))$ forms a non-Abelian group. Determine the order of each element in the group.

(c) Show that the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0$$

can be written as

$$\frac{\partial F}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial F}{\partial y'} - F\right) = 0,$$

where $F = F(y, y', x)$.

3. Find the extremal of the following functional

$$I = \int_0^1 (y'^2 + z'^2 - 4xz' - 4z)\, dx,$$

subjected to the constraint

$$\int_0^1 (y'^2 - xy' - z'^2)\, dx = 2,$$

given that $y(0) = 0$, $z(0) = 0$, $y(1) = 1$, and $z(1) = 1$. Calculate the corresponding value of the integral I.

4. (a) i. Represent the following function as an exponential Fourier transform

$$f(x) = \begin{cases} \sin x, & 0 < x < \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Hint: Use Fourier's inversion theorem.

ii. Show that your result can be written as

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos kx + \cos k(x - \pi)}{1 - k^2}\, dk.$$

(b) Use the method of tensor to establish the following vector identity,

$$\text{grad}\, \frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{u}) = \boldsymbol{u} \times \text{curl}\, \boldsymbol{u} + (\boldsymbol{u} \cdot \text{grad})\boldsymbol{u}.$$

LHS

Exam Formulae Sheet

$$\mathcal{L}[t^n] = n!/s^{n+1}$$
$$\mathcal{L}[\cos bt] = s/(s^2+b^2)$$
$$\mathcal{L}[\sin bt] = b/(s^2+b^2)$$
$$\mathcal{L}[f(at)] = \frac{1}{a}\bar{f}(s/a)$$
$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(u)\,\mathrm{d}u$$
$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n \bar{f}(s)}{\mathrm{d}s^n}$$

where $\mathcal{L}$ is the Laplace transform operator and $\mathcal{L}[f(t)] = \bar{f}(s)$.

Fourier transforms:

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\,\mathrm{d}t$$
$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\,\mathrm{d}\omega.$$

$$\cos A\,\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$$
$$\sin A\,\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Euler-Lagrange equation for several independent variables:

$$\frac{\partial F}{\partial y} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_{x_i}}\right),$$

where y_{x_i} stands for $\partial y/\partial x_i$ and $F = F(y, \frac{\partial y}{\partial x_1}, \ldots \frac{\partial y}{\partial x_n}, x_1, \ldots x_n)$.

– END OF PAPER –