Question 1

(a) Find the inverse Laplace transform of the following functions

i. $\frac{3s+1}{(s-1)(s^2+1)}$,

ii. $\frac{2}{(s^2+1)^2}$

(b) Given

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2-x, & 1 \le x \le 2, \\ 0, & x \ge 2, \end{cases}$$

find the Fourier cosine transform of $f(x)$ and use it to write $f(x)$ as an integral. Hence evaluate

$$\int_0^\infty \frac{\cos^2 k \sin^2\left(\frac{k}{2}\right)}{k^2} dk.$$

a)

i)

$$\tilde{f}(s) = \frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{1}{s^2+1} - 2\frac{s}{s^2+1}$$

$$f(t) = 2e^t + \sin t - 2\cos t$$

ii)

$$\tilde{f}(s) = \frac{2}{(s^2+1)^2} = 2\left(\frac{1}{s^2+1}\right)\left(\frac{1}{s^2+1}\right)$$

$$f(t) = 2\int_0^t \sin u \sin(t-u)\, du$$

$$= \int_0^t \cos(t-2u) - \cos t\, du$$

$$= \left[-\frac{1}{2}\sin(t-u) + u\cos t\right]_0^t$$

$$= \sin t + t\cos t$$

b)

$$\tilde{f}(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos kx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cos kx\,dx + \int_1^2 (2-x)\cos kx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[x\left(\frac{\sin kx}{k}\right)\right]_0^1 - \int_0^1 \frac{\sin kx}{k}\,dx + \left[(2-x)\left(\frac{\sin kx}{k}\right)\right]_1^2 + \int_1^2 \frac{\sin kx}{k}\,dx\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[\frac{\cos kx}{k^2}\right]_0^1 - \left[\frac{\cos kx}{k^2}\right]_1^2\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{\cos k}{k^2} - \frac{1}{k^2} - \frac{\cos 2k}{k^2} + \frac{\cos k}{k^2}\right)$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{k^2}(2\cos k - \cos 2k - 1)$$

$$= \sqrt{\frac{2}{\pi}}\frac{2}{k^2}(\cos k - \cos^2 k)$$

$$= \sqrt{\frac{2}{\pi}}\frac{2}{k^2}\cos k\,(1-\cos k)$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \tilde{f}(k)\cos kx\,dk = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{2}{k^2}\cos k\,(1-\cos k)\cos kx\,dk$$

when $x = 1,\ f(1) = 1,$

$$1 = \frac{4}{\pi}\int_0^\infty \frac{\cos^2 k\,(1-\cos k)}{k^2}\,dk$$

$$= \frac{4}{\pi}\int_0^\infty \frac{\cos^2 k\left[1 - 1 + 2\sin^2\left(\frac{k}{2}\right)\right]}{k^2}\,dk$$

$$= \frac{8}{\pi}\int_0^\infty \frac{\cos^2 k\sin^2\left(\frac{k}{2}\right)}{k^2}\,dk$$

$$\therefore \int_0^\infty \frac{\cos^2 k\sin^2\left(\frac{k}{2}\right)}{k^2}\,dk = \frac{\pi}{8}$$

Question 2

In a certain system of units, the electromagnetic stress tensor M_{ij} is given by

$$M_{ij} = E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(E_k E_k + B_k B_k),$$

where the electric and magnetic fields, $\vec{E}$ and $\vec{B}$, are first-order tensors. Show that M_{ij} is a second-order tensor.

Consider a situation in which $|\vec{E}| = |\vec{B}|$, but the directions of $\vec{E}$ and $\vec{B}$ are not parallel. Show that $\vec{E} \pm \vec{B}$ are principal axes of the stress tensor and find the corresponding principal values. Determine the 3rd principal axis and its corresponding principal value.

$$M_{ij} = E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(E_k E_k + B_k B_k)$$

$$M'_{ij} = L_{im}E_m L_{jn}E_n + L_{im}B_m L_{jn}B_n - \frac{1}{2}L_{ip}L_{jq}\delta_{pq}(L_{kr}E_r L_{ks}E_s + L_{kr}B_r L_{ks}B_s)$$

$$= L_{im}L_{jn}(E_m E_n + B_m B_n) - \frac{1}{2}L_{ip}L_{jq}\delta_{pq}(\delta_{rs}E_r E_s + \delta_{rs}B_r B_s)$$

$$= L_{im}L_{jn}\left[E_m E_n + B_m B_n - \frac{1}{2}\delta_{mn}(E_r E_r + B_r B_r)\right]$$

$$= L_{im}L_{jn}M_{mn}$$

$\therefore$ M_{ij} is a 2nd order tensor.

We let $v_i = E_i \pm B_i$.

$$M_{ij}v_j = M_{ij}(E_j \pm B_j)$$

$$= E_i E_j(E_j \pm B_j) + B_i B_j(E_j \pm B_j) - \frac{1}{2}\delta_{ij}(E^2 + B^2)(E_j \pm B_j)$$

$$= E_i E^2 \pm E_i E_j B_j + B_i B_j E_j \pm B_i B^2 - \frac{1}{2}(E^2 + B^2)(E_i \pm B_i)$$

$$= E_i E^2 \pm B_i E^2 \pm E_i E_j B_j + B_i E_j B_j - \frac{1}{2}(2E^2)(E_i \pm B_i)$$

$$= (E_i \pm B_i)[E^2 \pm E_j B_j - E^2]$$

$$= \pm E_j B_j (E_i \pm B_i)$$

$$= \pm(\vec{E}\cdot\vec{B})v_i$$

This shows that $\vec{E} \pm \vec{B}$ are eigenvectors (principal axes) of M_{ij} with principal values $\pm(\vec{E} \cdot \vec{B})$.

The 3rd axis is orthogonal to both of these,

$$(\vec{E} + \vec{B}) \times (\vec{E} - \vec{B}) = 2(\vec{B} \times \vec{E})$$

The principal value for this can be deduced from the trace of M_{ij},

$$Tr(M_{ij}) = E^2 + B^2 - \frac{3}{2}(E^2 + B^2) = -E^2$$

Since the two principal values found previously are $\pm\vec{E} \cdot \vec{B}$ which sums to zero, the principal value for this 3rd principal axis is – E^2.

Question 3

According to Fermat's principle, a light ray travels in a medium from one point to another so that the time of travel given by

$$\int \frac{ds}{v},$$

where s is arc length and v is velocity, is a minimum. Show that the path of travel is given by

$$vy'' + (1 + y'^2)\frac{\partial v}{\partial y} - y'(1 + y'^2)\frac{\partial v}{\partial x} = 0,$$

where $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$. Solve the differential equation for $v = \frac{1}{y}$.

$$\int \frac{ds}{v} = \int \frac{1}{v}\sqrt{1 + y'^2}\, dx$$

$$F = \frac{1}{v}\sqrt{1 + y'^2}$$

$$\frac{d}{dx}\frac{\partial F}{\partial y'} = \frac{\partial F}{\partial y}$$

$$\frac{d}{dx}\left(\frac{1}{v}\frac{y'}{\sqrt{1 + y'^2}}\right) = -\frac{1}{v^2}\frac{\partial v}{\partial y}\sqrt{1 + y'^2}$$

$$\frac{y'}{\sqrt{1 + y'^2}}\left(-\frac{1}{v^2}\right)\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}y'\right) + \frac{1}{v}\left(\frac{y''}{\sqrt{1 + y'^2}} - \frac{y'^2 y''}{(1 + y'^2)^{\frac{3}{2}}}\right) = -\frac{1}{v^2}\frac{\partial v}{\partial y}\sqrt{1 + y'^2}$$

$$y'\left(-\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}y'\right) + vy''\left(1 - \frac{y'^2}{1 + y'^2}\right) + \frac{\partial v}{\partial y}(1 + y'^2) = 0$$

$$-y'\frac{\partial v}{\partial x}+\frac{vy''}{1+y'^2}+\frac{\partial v}{\partial y}=0$$

$$vy''+(1+y'^2)\frac{\partial v}{\partial y}-y'(1+y'^2)\frac{\partial v}{\partial x}=0\ [shown]$$

When $v=\frac{1}{y}$, $\frac{\partial}{\partial y}\left(\frac{1}{y}\right)=-\frac{1}{y^2}$, $\frac{\partial}{\partial x}\left(\frac{1}{y}\right)=0$

$$\frac{y''}{y}+\frac{1+y'^2}{y^2}=0$$

$$yy''+1+y'^2=0$$

$$\frac{d}{dx}(yy')=-1$$

$$yy'=-x+c$$

$$y\,dy=(-x+c)dx$$

$$\frac{y^2}{2}=-\frac{x^2}{2}+cx+d$$

$$y=\sqrt{-x^2+2cx+2d}$$

Question 4

(a) Show that if p is prime then the set of rational number pairs (a,b), excluding $(0,0)$, with multiplication defined by $(a,b)\bullet(c,d)=(e,f)$, where

$$(a+b\sqrt{p})(c+f\sqrt{p})=e+f\sqrt{p},$$

forms an Abelian group. Show further that the mapping $(a,b)\to(a,-b)$ is an automorphism.

(b) Show that $\begin{pmatrix} x_2^2 & x_1x_2 \\ x_1x_2 & x_1^2 \end{pmatrix}$ is not a Cartesian tensor of order 2.

a) Let a,b,c,d,e,f be rational numbers and p be a prime.

$$(a+b\sqrt{p})(c+d\sqrt{p})=(ac+bdp)+(ad+bc)\sqrt{p}$$

We see that $e=ac+bdp$ and $f=ad+bc$ are rational number, so it is closed;

$$\begin{aligned}(a,b)\bullet[(c,d)\bullet(e,f)] &= (a+b\sqrt{p})[(c+d\sqrt{p})(e+f\sqrt{p})]\\ &= [(a+b\sqrt{p})(c+d\sqrt{p})](e+f\sqrt{p})\\ &= [(a,b)\bullet(c,d)]\bullet(e,f)\end{aligned}$$

So it is associative;

$$a + b\sqrt{p} \times \frac{1}{a + b\sqrt{p}} = 1, \qquad \text{where}$$

$$\frac{1}{a + b\sqrt{p}} = \frac{a - b\sqrt{p}}{(a + b\sqrt{p})(a - b\sqrt{p})} = \frac{a}{a^2 - b^2 p} - \frac{b\sqrt{p}}{a^2 - b^2 p}, \qquad \text{and}$$

$$\frac{a}{a^2 - b^2 p}, -\frac{b}{a^2 - b^2 p} \text{ are rational.}$$

So the inverse exists;

The identity is 1, which is rational;

Finally $(a, b) \bullet (c, d) = (c, d) \bullet (a, b)$ which is a result of multiplication.

∴ It forms an Abelian group.

$(a, b) \to (a, -b)$

We let

$$(a, b)' = (a, -b) = a - b\sqrt{p}, \qquad (c, d)' = (c, -d) = c - d\sqrt{p}$$

Then

$$[(a, b) \bullet (c, d)]' = (ac + bdp, ad + bc)' = (ac + bdp, -ad - bc)$$

$$(a, b)' \bullet (c, d)' = (a, -b) \bullet (c, -d) = (ac + bdp, -ad - bc)$$

∴ It is an Automorphism.

b) Consider a rotation of the (unprimed) coordinate axes through an angle θ to give the new (primed) axes. Under this rotation,

$$x_1 \to x_1' = x_1 \cos\theta + x_2 \sin\theta$$

$$x_2 \to x_2' = -x_1 \sin\theta + x_2 \cos\theta$$

The transformation matrix of L_{ij} will be $L = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Now consider the transformation of the first element $v_{11} = x_2^2$.

$$v_{11}' = (x_2')^2 = (-x_1 \sin\theta + x_2 \cos\theta)^2 = x_1^2 \sin^2\theta - 2x_1 x_2 \sin\theta \cos\theta + x_2^2 \cos^2\theta$$

However, $v_{11}' = L_{1k} L_{1l} v_{kl} = x_1^2 \sin^2\theta + 2x_1 x_2 \sin\theta \cos\theta + x_2 \cos^2\theta$

which is not equal to the above equation.

∴ It is not a 2nd order Cartesian tensor.

Solutions provided by: **A/Prof Paul Lim** (Q1, Q2, Q4) and **John Soo** (Q3)