NATIONAL UNIVERSITY OF SINGAPORE

PC3274 – MATHEMATICAL METHODS IN PHYSICS 2

(Semester I: AY 2009-10)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains FOUR questions and comprises FOUR printed pages.
2. Answer any THREE questions.
3. All questions carry equal marks.
4. Answers to the questions are to be written in the answer books.
5. This is a CLOSED BOOK examination.
6. A table of constants is provided.
7. The last page contains a list of formulae.

1. (a) Find the inverse Laplace transform of the following functions

   i. $\dfrac{3s+1}{(s-1)(s^2+1)}$,

   ii. $\dfrac{2}{(s^2+1)^2}$.

   (b) Given

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2-x, & 1 \le x \le 2, \\ 0, & x \ge 2, \end{cases}$$

   find the Fourier cosine transform of $f(x)$ and use it to write $f(x)$ as an integral. Hence evaluate

$$\int_0^\infty \frac{\cos^2 k \sin^2(k/2)}{k^2}\, dk.$$

2. In a certain system of units, the electromagnetic stress tensor M_{ij} is given by

$$M_{ij} = E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(E_k E_k + B_k B_k),$$

   where the electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, are first-order tensors. Show that M_{ij} is a second-order tensor.

   Consider a situation in which $|\boldsymbol{E}| = |\boldsymbol{B}|$, but the directions of $\boldsymbol{E}$ and $\boldsymbol{B}$ are not parallel. Show that $\boldsymbol{E} \pm \boldsymbol{B}$ are principal axes of the stress tensor and find the corresponding principal values. Determine the third principal axis and its corresponding principal value.

3. According to Fermat's principle, a light ray travels in a medium from one point to another so that the time of travel given by

$$\int \frac{\mathrm{d}s}{v},$$

where s is arc length and v is velocity, is a minimum. Show that the path of travel is given by

$$v y'' + (1 + y'^2)\frac{\partial v}{\partial y} - y'(1 + y'^2)\frac{\partial v}{\partial x} = 0,$$

where $y' = \mathrm{d}y/\mathrm{d}x$ and $y'' = \mathrm{d}^2 y/\mathrm{d}x^2$. Solve the differential equation for $v = 1/y$.

4. (a) Show that if p is prime then the set of rational number pairs (a, b), excluding $(0, 0)$, with multiplication defined by $(a, b) \bullet (c, d) = (e, f)$, where $(a + b\sqrt{p})(c + d\sqrt{p}) = e + f\sqrt{p}$, forms an Abelian group. Show further that the mapping $(a, b) \to (a, -b)$ is an automorphism.

(b) Show that

$$\begin{pmatrix} x_2^2 & x_1 x_2 \\ x_1 x_2 & x_1^2 \end{pmatrix}$$

is not a Cartesian tensor of order 2.

LHS

Exam Formulae Sheet

$$\mathcal{L}[t^n] = n!/s^{n+1}$$
$$\mathcal{L}[\cos bt] = s/(s^2+b^2)$$
$$\mathcal{L}[\sin bt] = b/(s^2+b^2)$$
$$\mathcal{L}[f(at)] = \frac{1}{a}\bar{f}(s/a)$$
$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(u)\,\mathrm{d}u$$
$$\mathcal{L}\left[\int_0^t f(u)\,\mathrm{d}u\right] = \frac{1}{s}\bar{f}(s)$$
$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n \bar{f}(s)}{\mathrm{d}s^n}$$
$$\mathcal{L}\left[\int_0^t f(u)g(t-u)\,\mathrm{d}u\right] = \bar{f}(s)\bar{g}(s)$$

where $\mathcal{L}$ is the Laplace transform operator and $\mathcal{L}[f(t)] = \bar{f}(s)$.

Fourier cosine transform:

$$\tilde{f}_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos kx\,\mathrm{d}x,$$
$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \tilde{f}_c(k)\cos kx\,\mathrm{d}k.$$

$$\mathcal{F}\left[\int_{-\infty}^\infty f(u)g(x-u)\,\mathrm{d}u\right] = \sqrt{2\pi}\,\tilde{f}(k)\tilde{g}(k)$$

Euler-Lagrange equation for several independent variables:

$$\frac{\partial F}{\partial y} = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_{x_i}}\right),$$

where y_{x_i} stands for $\partial y/\partial x_i$ and $F = F(y, \frac{\partial y}{\partial x_1}, \ldots \frac{\partial y}{\partial x_n}, x_1, \ldots x_n)$.

$$\cos A\,\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$$
$$\sin A\,\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

– END OF PAPER –