Question 1

A rigid body consists of 4 particles of masses m, 2m, 3m, 4m respectively situated at the points $(a,a,a),(a,-a,-a),(-a,a,-a),(-a,-a,a)$ and connected together by a massless framework.

(a) Find the inertia tensor at the origin and show that the principal moments of inertia are $20ma^2$, $(20\pm 2\sqrt{5})ma^2$.

(b) Find the principal axes and verify that they are orthogonal.

$$I=\sum M\left(r^2\delta_{ij}-x_i x_j\right)$$

a) $I_{11}=I_{22}=I_{33}=(10m)2a^2=20ma^2$

$$I_{12}=I_{21}=\sum m(-xy)=m(-a^2+2a^2+3a^2-4a^2)=0$$

$$I_{13}=I_{31}=\sum m(-xz)=m(-a^2+2a^2-3a^2+4a^2)=0$$

$$I_{23}=I_{32}=m(-a^2-2a^2+3a^2+4a^2)=4ma^2$$

$$\therefore I=ma^2\begin{pmatrix}20&0&2\\0&20&4\\2&4&20\end{pmatrix}=2ma^2\begin{pmatrix}10&0&1\\0&10&2\\1&2&10\end{pmatrix}$$

$$\det\left(\vec{I}-\lambda\mathbb{1}\right)=\begin{vmatrix}10-\lambda&0&1\\0&10-\lambda&2\\1&2&10-\lambda\end{vmatrix}=(10-\lambda)[(10-\lambda)^2-5]=0$$

$$\lambda=10,10\pm\sqrt{5}$$

∴ the principle moments of inertia,

$$20ma^2,2ma^2\left(10\pm\sqrt{5}\right)$$

b) for $\lambda=10$,

$$\begin{pmatrix}0&0&1\\0&0&2\\1&2&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\vec{0},\qquad z=0,x=2y,\qquad the\ axis=\begin{pmatrix}2\\1\\0\end{pmatrix}\to\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\\0\end{pmatrix}$$

for $\lambda=10+\sqrt{5}$,

$$\begin{pmatrix}-\sqrt{5}&0&1\\0&-\sqrt{5}&2\\1&2&-\sqrt{5}\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\vec{0},$$

$$z = \sqrt{5}x, 2z = \sqrt{5}y, \qquad the\ axis = \begin{pmatrix} 1 \\ 2 \\ \sqrt{5} \end{pmatrix} \rightarrow \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 2 \\ \sqrt{5} \end{pmatrix}$$

for $\lambda = 10 + \sqrt{5}$,

$$z = -\sqrt{5}x, 2z = -\sqrt{5}y, \qquad the\ axis = \begin{pmatrix} 1 \\ 2 \\ -\sqrt{5} \end{pmatrix} \rightarrow \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 2 \\ -\sqrt{5} \end{pmatrix}$$

$$\frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 2 \\ \sqrt{5} \end{pmatrix} \times \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 2 \\ -\sqrt{5} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

and so they are orthogonal to each other.

Question 2

(a) Use the method of Laplace transform to evaluate the following integral

$$\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt.$$

(b) Find the Fourier cosine transform of

$$f(x) = \begin{cases} 0, & x \le -a \\ 2(x+a), & -a < x \le 0 \\ -2(x-a), & 0 < x < a \\ 0, & x \ge a \end{cases}$$

Hence, evaluate

$$\int_0^\infty \frac{1-\cos k}{k^2}\,dk.$$

a) Compare the definition of Laplace transform with the integral,

$$\tilde{f}(s) = \int_0^\infty f(t)e^{-st}\,dt, \qquad \text{and} \qquad \int_0^\infty \frac{\sin t}{t}e^{-t}\,dt$$

We have $f(t) = \frac{\sin t}{t},\ s = 1.$

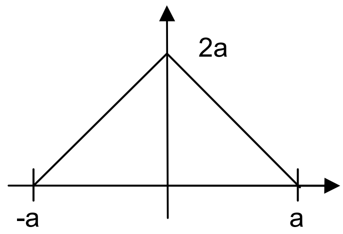
$$L[\sin t] = \frac{\omega}{s^2+\omega^2}$$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s'^2+1}\,ds' = [\tan^{-1} s']_s^\infty = \frac{\pi}{2} - \tan^{-1} s$$

Substitute $s = 1$,

$$\int_0^\infty \frac{\sin t}{t}e^{-t}\,dt = \frac{\pi}{2} - \tan^{-1} 1 = \frac{\pi}{4}$$

b)



Fourier cosine transform,

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos kx\,dx = \sqrt{\frac{2}{\pi}}\int_0^a -2(x-a)\cos kx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[-\frac{2}{k}(x-a)\sin kx\right]_0^a + \int_0^a \frac{2}{k}\sin kx\,dx\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{2}{k^2}\cos kx\right]_0^a$$

$$= \sqrt{\frac{8}{\pi}}\left(\frac{1-\cos ka}{k^2}\right)$$

$$f(x) = \frac{4}{\pi}\int_0^\infty \frac{1-\cos ka}{k^2}\cos kx\,dk$$

We let $a = 1, x = 0$,

$$f(0) = 2 = \frac{4}{\pi}\int_0^\infty \frac{1-\cos k}{k^2}dk$$

$$\therefore \int_0^\infty \frac{1-\cos k}{k^2}dk = \frac{\pi}{2}$$

Question 3

(a) Consider 2 sets $\mathcal{S}$ and $\mathcal{S}'$ defined as

$\mathcal{S} = \{1,2,3,4\}$ under multiplication (mod 5)

$\mathcal{S}' = \{1, i, -1, -i\}$ under ordinary multiplication of complex numbers where $i = \sqrt{-1}$. Show that $\mathcal{S}$ and $\mathcal{S}'$

i. each forms a group, and

ii. are isomorphic to each other.

(b) If x and y are 2 elements of any group, prove that xy and yx have the same order.

a) i) $\mathcal{S}$

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

ii) $\mathcal{S}'$

	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

Both $\mathcal{S}$ and $\mathcal{S}'$ are closed, multiplication associative, every element has its inverse, and the identity is in the group. ∴ they both form a group.

ii) we can set $\mathcal{S} \to \mathcal{S}'$ with elements $1 \to 1,\ 2 \to -i,\ 3 \to i,\ 4 \to -1$ and we find that $(XY)' = X'Y'$, one-to-one and onto. ∴ it is isomorphic to each other.

b) Since the groups are Abelian, we have $xy = yx = z$ for some z and therefore they should have the same order.

Question 4
Solve the Euler-Lagrange equation that makes the following integral stationary

$$I = \int_{x_0}^{x_1} (x^2 y'^2 + 2y^2 + 2xy)\, dx.$$

$$F = x^2 y'^2 + 2y^2 + 2xy$$

$$\frac{d}{dx}\frac{\partial F}{\partial y'} = \frac{\partial F}{\partial y}$$

$$\frac{d}{dx}(2x^2 y') = 4y + 2x$$

$$4xy' + 2x^2 y'' = 4y + 2x$$

$$x^2 y'' + 2xy' - 2y = x$$

By using substitution $x = e^u$,

$$\frac{dy}{dx} = \frac{1}{e^u}\frac{dy}{du}, \qquad \frac{d^2y}{dx^2} = \frac{1}{e^{2u}}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right)$$

$$e^{2u}\frac{1}{e^{2u}}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) + 2e^u\frac{1}{e^u}\frac{dy}{du} - 2y = e^u$$

$$\frac{d^2y}{du^2} + \frac{dy}{du} - 2y = e^u$$

auxiliary equation, $n^2 + n - 2 = 0 \quad \Rightarrow \quad n = 1, -2$

complimentary function, $y_h = Ae^u + Be^{-2u} = Ax + Bx^{-2}$

particular integral, $y_p = Cue^u$,

$y_p' = C(ue^u + e^u), \qquad y_p'' = C(2e^u + ue^u)$

$$C(2e^u + ue^u) + C(ue^u + e^u) - 2Cue^u = e^u \quad \Rightarrow \quad C = \frac{1}{3}$$

$$\therefore y = y_h + y_p = Ax + \frac{B}{x^2} + \frac{1}{3}ue^u = Ax + \frac{B}{x^2} + \frac{1}{3}x \ln x$$

Solutions provided by: **A/Prof Paul Lim** (Q1) and **John Soo** (Q2, Q3, Q4)