NATIONAL UNIVERSITY OF SINGAPORE

PC3274 – MATHEMATICAL METHODS IN PHYSICS 2

(Semester I: AY 2010-11)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains FOUR questions and comprises FOUR printed pages.
2. Answer ALL questions.
3. Each question carries equal marks.
4. Answers to the questions are to be written in the answer books.
5. This is a CLOSED BOOK examination.
6. The last page contains a list of formulae.

1. A rigid body consists of four particles of masses m, $2m$, $3m$, $4m$, respectively situated at the points (a, a, a), $(a, -a, -a)$, $(-a, a, -a)$, $(-a, -a, a)$ and connected together by a massless framework.

   (a) Find the inertia tensor at the origin and show that the principal moments of inertia are $20ma^2$, $(20 \pm 2\sqrt{5})ma^2$.

   (b) Find the principal axes and verify that they are orthogonal.

2. (a) Use the method of Laplace transform to evaluate the following integral

$$\int_0^\infty \frac{e^{-t} \sin t}{t} \, dt$$

   (b) Find the Fourier cosine transform of

$$f(x) = \begin{cases} 0, & x \leq -a \\ 2(x+a), & -a < x \leq 0 \\ -2(x-a), & 0 < x < a \\ 0, & x \geq a \end{cases}$$

   Hence, evaluate

$$\int_0^\infty \frac{1 - \cos k}{k^2} \, dk.$$

3. (a) Consider two sets $\mathcal{S}$ and $\mathcal{S}'$ defined as

$$\mathcal{S} = \{1, 2, 3, 4\} \quad \text{under multiplication (mod 5)}$$
$$\mathcal{S}' = \{1, i, -1, -i\} \quad \text{under ordinary multiplication of complex numbers}$$

where $i = \sqrt{-1}$. Show that $\mathcal{S}$ and $\mathcal{S}'$

i. each forms a group, and

ii. are isomorphic to each other.

(b) If x and y are two elements of any group, prove that xy and yx have the same order.

4. Solve the Euler-Lagrange equation that makes the following integral stationary

$$I = \int_{x_0}^{x_1} \left[x^2 y'^2 + 2y^2 + 2xy\right] \, \mathrm{d}x.$$

(Hint: Use the substitution $y = x^n$)

LHS

Exam Formulae Sheet

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, \mathrm{d}t, \quad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, \mathrm{d}\omega,$$

$$\mathcal{F}\left[\int_{-\infty}^{\infty} f(u) g(x-u)\, \mathrm{d}u\right] = \sqrt{2\pi}\, \tilde{f}(k)\tilde{g}(k)$$

Fourier sine transform : $\tilde{f}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t\, \mathrm{d}t, \; f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{f}(\omega) \sin \omega t\, \mathrm{d}\omega$

Fourier cosine transform : $\tilde{f}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t\, \mathrm{d}t, \; f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{f}(\omega) \cos \omega t\, \mathrm{d}\omega$

$f(t)$	$\bar{f}(s)$	s_0
c	c/s	0
ct^n	$cn!/s^{n+1}$	0
$\sin bt$	$b/(s^2+b^2)$	0
$\cos bt$	$s/(s^2+b^2)$	0
e^{at}	$1/(s-a)$	a
$t^n e^{at}$	$n!/(s-a)^{n+1}$	a
$\sinh at$	$a/(s^2-a^2)$	$\lvert a\rvert$
$\cosh at$	$s/(s^2-a^2)$	$\lvert a\rvert$
$e^{at}\sin bt$	$a/[(s-a)^2+b^2]$	a
$e^{at}\cos bt$	$(s-a)/[(s-a)^2+b^2]$	a

$$\mathcal{L}[f(at)] = \frac{1}{a}\bar{f}(s/a), \; \mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(u)\, \mathrm{d}u, \; \mathcal{L}\left[\int_0^t f(u)g(t-u)\, \mathrm{d}u\right] = \bar{f}(s)\bar{g}(s)$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n \bar{f}(s)}{\mathrm{d}s^n}, \; \mathcal{L}\left[\frac{\mathrm{d}^n f}{\mathrm{d}t^n}\right] = s^n \bar{f} - s^{n-1} f(0) - s^{n-2}\frac{\mathrm{d}f}{\mathrm{d}t}(0) \; \ldots - \frac{\mathrm{d}^{n-1} f}{\mathrm{d}t^{n-1}}(0)$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}, \; \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \; I_{ij} = \sum_{\alpha} m^{(\alpha)} \left[\left(r^{(\alpha)}\right)^2 \delta_{ij} - x_i^{(\alpha)} x_j^{(\alpha)}\right]$$

$$\frac{\partial F}{\partial y} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_{x_i}}\right), \text{ where } y_{x_i} = \partial y/\partial x_i, \text{ and } F = F\left(y, \frac{\partial y}{\partial x_1}, \ldots \frac{\partial y}{\partial x_n}, x_1, \ldots x_n\right).$$

$$\int \frac{1}{\sqrt{1-x^2}}\, \mathrm{d}x = \sin^{-1} x + c, \; -\int \frac{1}{\sqrt{1-x^2}}\, \mathrm{d}x = \cos^{-1} x + c, \; \int \frac{1}{1+x^2}\, \mathrm{d}x = \tan^{-1} x + c$$

$$\int \frac{1}{\sqrt{x^2+1}}\, \mathrm{d}x = \sinh^{-1} x + c, \; \int \frac{1}{\sqrt{x^2-1}}\, \mathrm{d}x = \cosh^{-1} x + c, \; \int \frac{1}{1-x^2}\, \mathrm{d}x = \tanh^{-1} x + c$$

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)], \; \sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

– END OF PAPER –