PC3274 Mathematical Methods in Physics II
AY2011/12 Solutions

Solutions provided by: Aren Tang

Question 1

$$x \bullet y = x + y + rxy$$

(a) Check for associativity:

$$\begin{aligned}
&x \bullet (y \bullet z) \\
=&x \bullet (y + z + ryz) \\
=&x + y + z + rxy + ryz + rxz + r^2xyz \\
=&(x + y + rxy) + rz(x + y + rxy) \\
=&(x \bullet y) \bullet z
\end{aligned}$$

(b)

$$\begin{aligned}
x \bullet y &= -\frac{1}{r} \\
x + y + rxy &= -\frac{1}{r} \\
y(1 + rx) + x &= -\frac{1}{r}
\end{aligned}$$

$$y = \frac{-(x + \frac{1}{r})}{(1 + rx)} \quad \text{and} \quad x = \frac{-(y + \frac{1}{r})}{(1 + ry)}$$

so, when $x = -\frac{1}{r}$ and $y = -\frac{1}{r}$, the other variable is undefined (it can be anything). So, $x \bullet y = -\frac{1}{r}$ is true if and only if x or y has a value of $-\frac{1}{r}$.

(c) Associativity has been proven in part (a). There exists and Identity, where $I = 0$.

Closure:

$$x \bullet y = x + y + rxy$$

Since x, y, r are all real, $x \bullet y$ is also real. Also, since $x \neq -\frac{1}{r}, y \neq -\frac{1}{r}$, $x \bullet y \neq -\frac{1}{r}$. $x \bullet y$ belongs to the set.

Inverse:

$$\begin{aligned}
x \bullet x^{-1} &= I \\
x + xx^{-1} + rxx^{-1} &= 0 \\
x^{-1} &= -\frac{x}{1 + rx}
\end{aligned}$$

since x is real and r is real, x^{-1} is also real. The inverse of x exists in this set.

Question 2

(a)We know that,

$$\epsilon_{ijk}\epsilon_{nlm} = \begin{vmatrix} \delta_{in} & \delta_{il} & \delta_{im} \\ \delta_{jn} & \delta_{jl} & \delta_{jm} \\ \delta_{kn} & \delta_{kl} & \delta_{km} \end{vmatrix}$$

By setting n=k, we reduce the equation to:

$$\epsilon_{ijk}\epsilon_{klm} = \begin{vmatrix} 0 & \delta_{il} & \delta_{im} \\ 0 & \delta_{jl} & \delta_{jm} \\ 1 & \delta_{kl} & \delta_{km} \end{vmatrix} = \begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix}$$

Thus, we get:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

(b)

$$I_{ij} = \int \rho(x_i x_j \delta_{ij} - x_i x_j)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

where $\rho = M/L^3$.
From symmetry, we can see that:
$I_{11} = I_{22} = I_{33}$;
$I_{12} = I_{21}$, $I_{13} = I_{31}$, $I_{23} = I_{32}$.
Furthermore, since the integration limits are all the same, $I_{12} = I_{13} = I_{23}$, leaving us with two independant components.

$$\begin{aligned} I_{11} &= \int \rho(y^2 + z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \rho L \int (y^2 + z^2)\,\mathrm{d}y\mathrm{d}z \\ &= \rho L^2 \int (\frac{L^2}{3} + z^2)\,\mathrm{d}z \\ &= \frac{2\rho L^5}{3} = \frac{2ML^2}{3} \end{aligned}$$

$$\begin{aligned} I_{12} &= -\rho \int xy\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= -\rho L \int xy\,\mathrm{d}x\mathrm{d}y \\ &= -\rho \frac{L^3}{2} \int y^2\,\mathrm{d}y \\ &= -\rho \frac{L^5}{4} = -\frac{1}{4}ML^2 \end{aligned}$$

Thus we have:

$$I = ML^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$$

Question 3

$$I = \int_{x_0}^{x_1} (y^2 - y'^2 - 2y\sin x)\,\mathrm{d}x$$

With, $F = y^2 - y'^2 - 2y\sin x$ we use the Euler-Lagrange Equation:

$$\frac{\partial F}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}$$

$$\begin{aligned} 2y - 2\sin x &= -2\frac{\mathrm{d}}{\mathrm{d}x}y' \\ y'' + y &= \sin x \end{aligned}$$

From this differential equation, we get the homogenous solution:

$$y_1 = Ae^{-ix} + Be^{ix}$$

Solving the non-homogenous part will require a trial function, we try with:

$$\begin{aligned} y_2 &= \mathrm{Re}[Cxe^{ix}] \\ y_2' &= \mathrm{Re}[iCxe^{ix} + Ce^{ix}] \\ y_2'' &= \mathrm{Re}[-Cxe^{ix} + iCe^{ix} + iCe^{ix}] \end{aligned}$$

Substituting these into the differential equation above:

$$\begin{aligned} \mathrm{Re}[2iCe^{ix}] &= \sin x \\ 2iCi\sin x &= \sin x \\ -2C &= 1 \\ C &= -\frac{1}{2} \end{aligned}$$

So, we have:

$$\begin{aligned} y_2 &= \mathrm{Re}[-\frac{1}{2}xe^{-ix}] \\ y_2 &= -\frac{1}{2}x\cos x \end{aligned}$$

And since $y = y_1 + y_2$,

$$y = Ae^{-x} + Be^{x} - \frac{1}{2}x\cos x$$

Question 4

(a)

$$
\begin{aligned}
y'' - 4y' + 5y &= 2e^{-2x}\cos x \\
s^2\bar{y} - sy(0) - y'(0) + 4(sy - y(0)) + 5\bar{y} &= 2\left[\frac{s+2}{(s+2)^2+1}\right] \\
(s^2 + 4s + 5)\bar{y} &= \frac{2(s+2)}{(s+2)^2+1} + 3 \\
((s+2)^2 + 1)\bar{y} &= \frac{2(s+2)}{(s+2)^2+1} + 3 \\
y &= L^{-1}\left[\frac{2(s+2)}{((s+2)^2+1)^2}\right] + 3e^{-2x}\sin x
\end{aligned}
$$

Evaluating the first term:

$$
\begin{aligned}
L^{-1}\left[\frac{2(s+2)}{((s+2)^2+1)^2}\right] &= L^{-1}\left[\frac{(s+2)}{(s+2)^2+1} \times \frac{2}{(s+2)^2+1}\right] \\
&= L^{-1}[\bar{f}\bar{g}] \\
&= \int_0^x f(x')g(x-x')\,\mathrm{d}x'
\end{aligned}
$$

Where:

$$
\begin{aligned}
f(x) &= e^{-2x}\cos x \\
g(x) &= 2e^{-2x}\sin x
\end{aligned}
$$

So,

$$
\begin{aligned}
&\int_0^x f(x')g(x-x')\,\mathrm{d}x' \\
=&\int_0^x e^{-2x'}\cos x' 2e^{-2(x-x')}\sin(x-x')\,\mathrm{d}x' \\
=&e^{-2x}x\sin x
\end{aligned}
$$

Substituting this back into the equation for y above:

$$
y = e^{-2x}(x\sin x + 3\sin x)
$$

(b)

$$
\begin{aligned}
\tilde{f}(k) &= \sqrt{\frac{2}{\pi}} \int_0^{\pi/2} \cos x \cos kx \, \mathrm{d}x \\
&= -\sqrt{\frac{2}{\pi}} \frac{\cos\left(\frac{k\pi}{2}\right)}{k^2 - 1} \\
|\tilde{f}(k)|^2 &= \frac{2}{\pi} \frac{\cos^2\left(\frac{k\pi}{2}\right)}{(k^2-1)^2}
\end{aligned}
$$

From Parseval's Theorem,

$$
\begin{aligned}
\int_0^{\infty} \frac{\cos^2\left(\frac{k\pi}{2}\right)}{(k^2-1)^2} \, \mathrm{d}k &= \frac{\pi}{2} \int_0^{\infty} \cos^2 x \, \mathrm{d}x \\
&= \frac{\pi}{2} \int_0^{\pi/2} \cos^2 x \, \mathrm{d}x \\
&= \frac{\pi^2}{8}
\end{aligned}
$$