# NATIONAL UNIVERSITY OF SINGAPORE

# PC3274 – MATHEMATICAL METHODS IN PHYSICS 2

(Semester I: AY 2011-12)

Time Allowed: 2 Hours

---

**INSTRUCTIONS TO CANDIDATES**

1. This examination paper contains FOUR questions and comprises FOUR printed pages.
2. Answer **ALL** questions.
3. Each question carries equal marks.
4. Answers to the questions are to be written in the answer books.
5. This is a CLOSED BOOK examination.
6. The last page contains a list of formulae.

1. A binary operation $\bullet$ on the set of real numbers is defined by

$$x \bullet y = x + y + rxy,$$

where $r$ is a non-zero real number.

(a) Show that the operation $\bullet$ is associative.

(b) Prove that $x \bullet y = -\frac{1}{r}$ if, and only if, $x = -\frac{1}{r}$ or $y = -\frac{1}{r}$.

(c) Prove that the set of all real numbers excluding $-\frac{1}{r}$ forms a group under the operation $\bullet$.

2. (a) Prove the tensor identity

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl},$$

where $\epsilon_{ijk}$ is the Levi-Civita symbol and $\delta_{ij}$ the Kronecker delta.

(b) Calculate the moment of inertia tensor for a solid cube of side length $L$ and mass $M$ about one of its corners.

3. Find the function $y(x)$ that makes the following integral stationary

$$I = \int_{x_0}^{x_1} (y^2 - y'^2 - 2y \sin x)\, \mathrm{d}x \ .$$

4. (a) Use the method of Laplace transform to solve the second-order differential equation

$$y'' + 4y' + 5y = 2e^{-2x} \cos x,$$

subject to the initial conditions $y(0) = 0$ and $y'(0) = 3$.

(b) Find the Fourier transform of

$$f(x) = \begin{cases} \cos x, & -\pi/2 < x < \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$$

Hence, using the Parseval's theorem, namely,

$$\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \, \mathrm{d}k,$$

evaluate

$$\int_0^{\infty} \frac{\cos^2(k\pi/2)}{(1-k^2)^2} \, \mathrm{d}k \ .$$

LHS

## Formulae Sheet

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, \mathrm{d}t, \quad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, \mathrm{d}\omega,$$

Fourier sine transform : $\tilde{f}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t\, \mathrm{d}t, \; f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{f}(\omega) \sin \omega t\, \mathrm{d}\omega$

Fourier cosine transform : $\tilde{f}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t\, \mathrm{d}t, \; f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{f}(\omega) \cos \omega t\, \mathrm{d}\omega$

$$\mathcal{F}\left[\int_{-\infty}^{\infty} f(u) g(x-u)\, \mathrm{d}u\right] = \sqrt{2\pi}\, \tilde{f}(k) \tilde{g}(k), \quad \mathcal{F}[f(x)g(x)] = \frac{1}{\sqrt{2\pi}} \tilde{f}(k) * \tilde{g}(k)$$

| $f(t)$ | $\bar{f}(s)$ | $s_0$ |
|---|---|---|
| $c$ | $c/s$ | $0$ |
| $ct^n$ | $cn!/s^{n+1}$ | $0$ |
| $\sin bt$ | $b/(s^2+b^2)$ | $0$ |
| $\cos bt$ | $s/(s^2+b^2)$ | $0$ |
| $e^{at}$ | $1/(s-a)$ | $a$ |
| $t^n e^{at}$ | $n!/(s-a)^{n+1}$ | $a$ |
| $\sinh at$ | $a/(s^2-a^2)$ | $\|a\|$ |
| $\cosh at$ | $s/(s^2-a^2)$ | $\|a\|$ |
| $e^{at} \sin bt$ | $b/[(s-a)^2+b^2]$ | $a$ |
| $e^{at} \cos bt$ | $(s-a)/[(s-a)^2+b^2]$ | $a$ |

$$\mathcal{L}[f(at)] = \frac{1}{a} \bar{f}(s/a), \; \mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(u)\, \mathrm{d}u, \; \mathcal{L}\left[\int_0^t f(u) g(t-u)\, \mathrm{d}u\right] = \bar{f}(s) \bar{g}(s)$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n \bar{f}(s)}{\mathrm{d}s^n}, \; \mathcal{L}\left[\frac{\mathrm{d}^n f}{\mathrm{d}t^n}\right] = s^n \bar{f} - s^{n-1} f(0) - s^{n-2} \frac{\mathrm{d}f}{\mathrm{d}t}(0) \ldots - \frac{\mathrm{d}^{n-1} f}{\mathrm{d}t^{n-1}}(0)$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}, \; \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \; I_{ij} = \sum_{\alpha} m^{(\alpha)} \left[\left(r^{(\alpha)}\right)^2 \delta_{ij} - x_i^{(\alpha)} x_j^{(\alpha)}\right]$$

$$\frac{\partial F}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right), \; \frac{\partial F}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(y' \frac{\partial F}{\partial y'} - F\right) = 0, \; \frac{\partial F}{\partial y} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_{x_i}}\right), \text{ where } y_{x_i} = \partial y / \partial x_i$$

$$\int \frac{1}{\sqrt{1-x^2}}\, \mathrm{d}x = \sin^{-1} x + c, \; -\int \frac{1}{\sqrt{1-x^2}}\, \mathrm{d}x = \cos^{-1} x + c, \; \int \frac{1}{1+x^2}\, \mathrm{d}x = \tan^{-1} x + c$$

$$\int \frac{1}{\sqrt{x^2+1}}\, \mathrm{d}x = \sinh^{-1} x + c, \; \int \frac{1}{\sqrt{x^2-1}}\, \mathrm{d}x = \cosh^{-1} x + c, \; \int \frac{1}{1-x^2}\, \mathrm{d}x = \tanh^{-1} x + c$$

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)], \; \sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B), \; \sin A - \sin B = 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)$$

$$\cos A + \cos B = 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$$

– END OF PAPER –