PC3274 Mathematical Methods in Physics II

2017/2018 Examination Model Answers

1. (a)

$$\begin{aligned} 1^i &= \exp\left[i \,\mathrm{Ln}\,(1)\right] \\ &= \exp\left[i(\ln 1 + i0 + 2\pi i n)\right] \\ &= \exp\left(-2\pi n\right) \end{aligned}$$

where n is any integer. These points lie along the positive real axis.

(b)(i) There is one order-2 pole at $z = 0$, and two simple (order-1) poles at $z = 2$ and $z = \frac{1}{2}$.

Calculate residues:

$$\begin{aligned} \mathrm{Res}\big(0\big) &= \lim_{z\to 0} \frac{\mathrm{d}}{\mathrm{d}z}\left[z^2 f(z)\right] \\ &= \lim_{z\to 0} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{i(z^4+1)}{2(z-2)(2z-1)}\right] \\ &= \lim_{z\to 0}\left[\frac{4iz^3}{2(z-2)(2z-1)} - \frac{i(z^4+1)}{2(z-2)^2(2z-1)} - \frac{2i(z^4+1)}{2(z-2)(2z-1)^2}\right] \\ &= \frac{5i}{8} \end{aligned}$$

$$\begin{aligned} \mathrm{Res}\big(2\big) &= \lim_{z\to 2}\left[\big(z-2\big)f(z)\right] \\ &= \lim_{z\to 2}\left[\frac{i(z^4+1)}{2z^2(2z-1)}\right] \\ &= \frac{17i}{24} \end{aligned}$$

$$\begin{aligned} \mathrm{Res}\big(\tfrac{1}{2}\big) &= \lim_{z\to \frac{1}{2}}\left[\big(z-\tfrac{1}{2}\big)f(z)\right] \\ &= \lim_{z\to \frac{1}{2}}\left[\frac{i(z^4+1)}{4z^2(z-2)}\right] \\ &= -\frac{17i}{24} \end{aligned}$$

(ii) Since $z = \mathrm{e}^{i\theta}$, we have

$$\begin{aligned} \cos\theta &= \tfrac{1}{2}(z + z^{-1}) \\ \cos 2\theta &= \tfrac{1}{2}(z^2 + z^{-2}) \end{aligned}$$

Also, $-i\frac{\mathrm{d}z}{z} = \mathrm{d}\theta$. Denoting the unit circle by C, we have

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta}\,\mathrm{d}\theta &= \int_C \frac{\frac{1}{2}(z^2 + z^{-2})}{5 - 2(z + z^{-1})}\,\frac{-i\,\mathrm{d}z}{z} \\
&= \int_C \frac{-i(z^4 + 1)}{2z^2[5z - 2(z^2 + 1)]}\,\mathrm{d}z \\
&= \int_C \frac{i(z^4 + 1)}{2z^2(2z^2 - 5z + 2)}\,\mathrm{d}z \\
&= \int_C \frac{i(z^4 + 1)}{2z^2(z - 2)(2z - 1)}\,\mathrm{d}z \\
&= \int_C f(z)\,\mathrm{d}z
\end{aligned}$$

Since C encloses the poles at $z = 0$ and $z = \frac{1}{2}$, by the residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i\big[\mathrm{Res}(0) + \mathrm{Res}\big(\tfrac{1}{2}\big)\big]$$

Hence

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta}\,\mathrm{d}\theta = 2\pi i\left(\frac{5i}{8} - \frac{17i}{24}\right) = \frac{\pi}{6}$$

2. (a) Length of an infinitesimal segment along the path:

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + y'^2}\,\mathrm{d}x$$

Time taken for particle to traverse this infinitesimal segment:

$$\mathrm{d}t = \frac{\mathrm{d}s}{v(y)} = \frac{\sqrt{1 + y'^2}}{v(y)}\,\mathrm{d}x$$

Total time taken:

$$I[y(x)] = \int \mathrm{d}t = \int_0^{x_0} \frac{\sqrt{1 + y'^2}}{v(y)}\,\mathrm{d}x$$

Thus

$$F(y, y') = \frac{\sqrt{1 + y'^2}}{v(y)}$$

Since F does not depend on x explicitly, the Euler–Lagrange equation becomes the Beltrami identity:

$$F - y'\frac{\partial F}{\partial y'} = \frac{\sqrt{1 + y'^2}}{v(y)} - \frac{y'^2}{v(y)\sqrt{1 + y'^2}} = \text{const}$$

Call this constant $\frac{1}{k}$, so

$$\frac{1}{v(y)\sqrt{1+y'^2}} = \frac{1}{k}$$

Rearranging:

$$y'^2 = \frac{k^2 - v(y)^2}{v(y)^2}$$

Thus

$$y' = \pm\sqrt{\frac{k^2 - v(y)^2}{v(y)^2}}$$

or

$$\mathrm{d}x = \pm\sqrt{\frac{v(y)^2}{k^2 - v(y)^2}}\,\mathrm{d}y$$

(b) Let $v(y) = ay$. Change variable $y = \frac{k}{a}\sin z$, $\mathrm{d}y = \frac{k}{a}\cos z\,\mathrm{d}z$, so that

$$\begin{aligned}\mathrm{d}x &= \pm\sqrt{\frac{a^2y^2}{k^2 - a^2y^2}}\,\mathrm{d}y \\ &= \pm\sqrt{\frac{k^2\sin^2 z}{k^2 - k^2\sin^2 z}}\,\frac{k}{a}\cos z\,\mathrm{d}z \\ &= \pm\frac{k}{a}\sin z\,\mathrm{d}z\end{aligned}$$

Integrate:

$$x + c = \mp\frac{k}{a}\cos z$$

where c is a constant.

Hence

$$\begin{aligned}y(x) &= \frac{k}{a}\sqrt{1-\cos^2 z} \\ &= \sqrt{\frac{k^2}{a^2} - \frac{k^2}{a^2}\cos^2 z} \\ &= \sqrt{\frac{k^2}{a^2} - (x+c)^2}\end{aligned}$$

which describes a circle of radius k/a and centre at $(-c, 0)$.

Impose BCs:

At $x = 0$:

$$y(0) = \sqrt{\frac{k^2}{a^2} - c^2} = 0$$

which implies that

$$c = \pm \frac{k}{a}$$

As $x = x_0$ is a variable endpoint, we impose

$$y'(x) = -\frac{x + c}{\sqrt{\frac{k^2}{a^2} - (x + c)^2}} = 0$$

at this point. This implies that

$$c = -x_0$$

and

$$\frac{k^2}{a^2} = x_0^2$$

The solution is thus

$$y(x) = \sqrt{x_0^2 - (x - x_0)^2}$$

3. (a) Group multiplication table:

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

It is an Abelian group, since the table is symmetric about the diagonal.

(b) Since $\mathcal{G}$ is Abelian, it has 4 conjugacy classes. This is equal to the number N of inequivalent irreps that $\mathcal{G}$ has. Moreover, $\mathcal{G}$ is a group of order $g = 4$. By the summation rule for irreps,

$$\sum_i^4 n_i^2 = 4$$

The only way that this can be satisfied is if $n_1 = n_2 = n_3 = n_4 = 1$, i.e., $\mathcal{G}$ has 4 one-dimensional irreps. In particular, it cannot have any two-dimensional irreps.

(c) The first irrep is just the trivial representation:

$$\{\mathsf{I}, \mathsf{A}, \mathsf{B}, \mathsf{C}\} \mapsto \{1, 1, 1, 1\}$$

Since $\mathsf{A}^2 = \mathsf{I}$, we have

$$\mathsf{D}(\mathsf{A})^2 = \mathsf{D}(\mathsf{I}) = 1$$

i.e.,

$$\mathsf{D}(\mathsf{A}) = \pm 1$$

Similarly, $\mathsf{B}^2 = \mathsf{I}$ and $\mathsf{C}^2 = \mathsf{I}$ imply that

$$\mathsf{D}(\mathsf{B}) = \pm 1\,, \qquad \mathsf{D}(\mathsf{C}) = \pm 1$$

Suppose $\mathsf{D}(\mathsf{A}) = +1$. From $\mathsf{AB} = \mathsf{C}$, we see that $\mathsf{D}(\mathsf{A})\mathsf{D}(\mathsf{B}) = \mathsf{D}(\mathsf{C})$, i.e., $\mathsf{D}(\mathsf{B}) = \mathsf{D}(\mathsf{C})$. On the other hand, if $\mathsf{D}(\mathsf{A}) = -1$, then $\mathsf{D}(\mathsf{B}) = -\mathsf{D}(\mathsf{C})$. Thus, the other 3 irreps are

$$\{\mathsf{I}, \mathsf{A}, \mathsf{B}, \mathsf{C}\} \mapsto \{1, 1, -1, -1\}$$

$$\{\mathsf{I}, \mathsf{A}, \mathsf{B}, \mathsf{C}\} \mapsto \{1, -1, 1, -1\}$$

$$\{\mathsf{I}, \mathsf{A}, \mathsf{B}, \mathsf{C}\} \mapsto \{1, -1, -1, 1\}$$

Character table:

	$\{\mathsf{I}\}$	$\{\mathsf{A}\}$	$\{\mathsf{B}\}$	$\{\mathsf{C}\}$
A_1	1	1	1	1
A_2	1	1	−1	−1
A_3	1	−1	1	−1
A_4	1	−1	−1	1
D	2	−2	0	0

where in the last row we have listed the characters for the representation given in the question. It can be seen that

$$\mathsf{D} = \mathsf{A}_3 \oplus \mathsf{A}_4$$

Thus the diagonal form of the representative matrices are

$$\mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathsf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathsf{C} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

4. (a) Since we have $\mathsf{A}' = \mathsf{LA}$ and $\mathsf{B}' = \mathsf{LB}$, this implies L must have the form

$$\mathsf{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & a \\ 0 & 0 & b \\ -1 & 1 & c \end{pmatrix}$$

We determine a, b and c by requiring that L is orthogonal and satisfies $\det(\mathsf{L}) = +1$

$$\mathsf{L}\mathsf{L}^{\mathrm{T}} = \frac{1}{2}\begin{pmatrix} 1 & 1 & a \\ 0 & 0 & b \\ -1 & 1 & c \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ a & b & c \end{pmatrix} = \mathsf{I}$$

giving $a = 0$, $b = \pm\sqrt{2}$ and $c = 0$. The determinant of L is $\frac{1}{2\sqrt{2}}(-b - b + 0)$, thus requiring that $b = -\sqrt{2}$. Hence

$$\mathsf{L} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -\sqrt{2} \\ -1 & 1 & 0 \end{pmatrix}$$

Now the transformation law for a second-order tensor $T'_{ij} = L_{ik}L_{jl}T_{kl}$ is equivalent to the matrix product $\mathsf{T}' = \mathsf{L}\mathsf{T}\mathsf{L}^{\mathrm{T}}$. We have

$$\begin{aligned}
\mathsf{T}' &= \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -\sqrt{2} \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & -\sqrt{2} & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -2\sqrt{2} \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & -\sqrt{2} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

(b) In this case, L has the form

$$\mathsf{L} = \begin{pmatrix} 0 & a & d \\ 0 & b & e \\ 1 & c & f \end{pmatrix}$$

Now,

$$\mathsf{L}\mathsf{L}^{\mathrm{T}} = \begin{pmatrix} 0 & a & d \\ 0 & b & e \\ 1 & c & f \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} a^2 + d^2 & ab + de & ac + df \\ ab + de & b^2 + e^2 & bc + ef \\ ac + df & bc + ef & 1 + c^2 + f^2 \end{pmatrix} = \mathsf{I}$$

The last component (the 33-component) implies that $c = f = 0$. The remaining components give:

$$a^2 + d^2 = 1 \tag{1}$$

$$ab + de = 0 \tag{2}$$

$$b^2 + e^2 = 1 \tag{3}$$

(These equations can be solved, but we don't need to). This also turns L into the form

$$\mathsf{L} = \begin{pmatrix} 0 & a & d \\ 0 & b & e \\ 1 & 0 & 0 \end{pmatrix}$$

Thus we have

$$\begin{aligned} \mathsf{T}' &= \begin{pmatrix} 0 & a & d \\ 0 & b & e \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ a & b & 0 \\ d & e & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & a & 2d \\ 0 & b & 2e \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ a & b & 0 \\ d & e & 0 \end{pmatrix} \\ &= \begin{pmatrix} a^2 + 2d^2 & ab + 2de & 0 \\ ab + 2de & b^2 + 2e^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Using Eqs. (1)–(3), we have

$$\mathsf{T}' = \begin{pmatrix} 1 + d^2 & de & 0 \\ de & 1 + e^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since the first component is 1, this means that $d = 0$. From Eq. (1), $a = \pm 1$. From Eq. (2), $ab = 0$, i.e., $b = 0$. From Eq. (3), $e = \pm 1$. Hence

$$\mathsf{T}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$