NATIONAL UNIVERSITY OF SINGAPORE

PC3274 MATHEMATICAL METHODS IN PHYSICS II

(Semester I: AY 2017–18)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Write your matriculation number only. **Do not write your name.**
2. This examination paper contains **FOUR** questions and comprises **FOUR** printed pages.
3. Answer **ALL** questions.
4. Please start each question on a new page.
5. This is a **closed book** examination.
6. Only non-programmable calculators are allowed.
7. A list of formulae is attached at the end of the paper.

1. **Complex Variables and Complex Integration.**

(a) Find the values of

$$1^i.$$

Where do these points lie in the complex plane?

(b) Consider the function

$$f(z) = \frac{i(z^4+1)}{2z^2(z-2)(2z-1)}.$$

(i) Identify all the poles of $f(z)$. Find the order and residue of each pole.

(ii) By considering the integral of $f(z)$ around the unit circle $z = \mathrm{e}^{i\theta}$, evaluate the integral

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta}\, \mathrm{d}\theta\,.$$

[30 marks]

2. **Calculus of Variations.** A particle moves through a certain field in the (x, y)-plane, with a speed $v(y)$ that depends on y.

(a) Find a functional $I[y(x)]$ for the total time taken for the particle to travel along a path $y(x)$, where $0 \le x \le x_0$. Extremise $I[y(x)]$, and show that the path of the particle satisfies

$$y' = \pm\sqrt{\frac{k^2 - v(y)^2}{v(y)^2}}\,, \tag{1}$$

where $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$ and k is an arbitrary constant.

(b) Suppose $v(y) = ay$, where a is a positive constant and $y \ge 0$. By performing the change of variable $y = \frac{k}{a}\sin z$, integrate Eq. (1) and find the path of the particle that starts from the point $(0, 0)$ and ends somewhere on the line $x = x_0$.

[20 marks]

3. **Group Theory and Representation Theory.** Consider the following set of four 2×2 matrices:

$$\mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathsf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}, \quad \mathsf{C} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}.$$

(a) Verify that they form a group $\mathcal{G}$ under matrix multiplication, by constructing their multiplication table. Is this group Abelian?

(b) How many conjugacy classes does $\mathcal{G}$ have? Hence use the summation rule for irreducible representations (irreps) to explain why the above set of 2×2 matrices does not form an irrep of $\mathcal{G}$.

(c) Find all the irreps of $\mathcal{G}$, and write down the associated character table. Use this table to decompose the above 2×2 matrices as a direct sum of irreps of $\mathcal{G}$, and hence find the diagonal form of these matrices.

[30 marks]

4. **Tensors.** The components of two vectors, $\boldsymbol{A}$ and $\boldsymbol{B}$, and a second-order tensor, $\boldsymbol{T}$, are given in a coordinate system by

$$\mathsf{A} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathsf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) In another coordinate system, obtained from the original one by a proper rotation, the components of $\boldsymbol{A}$ and $\boldsymbol{B}$ are

$$\mathsf{A}' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathsf{B}' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Find the components of $\boldsymbol{T}$ in this coordinate system.

(b) In yet another coordinate system, also obtained from the original one by a proper rotation, the components of $\boldsymbol{A}$ are

$$\mathsf{A}' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

The components of $\boldsymbol{B}$ are unknown in this coordinate system. However, the first component of $\boldsymbol{T}$ is known to have value 1 in this coordinate system, i.e.,

$$\mathsf{T}' = \begin{pmatrix} 1 & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} .$$

Find the rest of the components of $\boldsymbol{T}$ in this coordinate system.

[20 marks]

(ET)

– END OF PAPER –

PC3274 List of Formulae

- Common integrals:

$$\int x^\alpha \, \mathrm{d}x = \begin{cases} \frac{x^{\alpha+1}}{\alpha+1} & (\alpha \text{ real}, \alpha \neq -1) \\ \ln x & (\alpha = -1) \end{cases}$$

$$\int \sin x \, \mathrm{d}x = -\cos x$$

$$\int \cos x \, \mathrm{d}x = \sin x$$

- Integration by parts:

$$\int_a^b f(x) g'(x) \, \mathrm{d}x = f(x) g(x) \Big|_a^b - \int_a^b f'(x) g(x) \, \mathrm{d}x$$

- Euler's formula:

$$\mathrm{e}^{i\theta} = \cos\theta + i \sin\theta$$

- Binomial theorem:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$$

- Cauchy–Riemann relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

- Complex exponential function:

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

- Complex trigonometric and hyperbolic functions:

$$\sin z = \tfrac{1}{2i}(\mathrm{e}^{iz} - \mathrm{e}^{-iz})$$
$$\cos z = \tfrac{1}{2}(\mathrm{e}^{iz} + \mathrm{e}^{-iz})$$
$$\sinh z = \tfrac{1}{2}(\mathrm{e}^{z} - \mathrm{e}^{-z})$$
$$\cosh z = \tfrac{1}{2}(\mathrm{e}^{z} + \mathrm{e}^{-z})$$

- Complex logarithm:

$$\mathrm{Ln}\, z = \ln r + i(\theta + 2n\pi)$$

- Complex exponent:

$$t^z = \exp(z \, \mathrm{Ln}\, t)$$

- Cauchy integral formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \, \mathrm{d}z$$

- Residue of a pole of order m:

$$\mathrm{Res}(z_0) = \lim_{z \to z_0} \left\{ \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} [(z-z_0)^m f(z)] \right\}$$

- Residue theorem:

$$\oint_C f(z) \, \mathrm{d}z = 2\pi i \sum_j \mathrm{Res}\,(z_j)$$

- Bromwich integral:

$$f(t) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \bar{f}(s) \, \mathrm{e}^{st} \, \mathrm{d}s, \qquad \lambda > 0$$

- Functional:

$$I[y(x)] = \int_a^b F(y(x), y'(x), x) \, \mathrm{d}x$$

- Euler–Lagrange equation:

$$\frac{\partial F}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right)$$

- Case that F does not contain y explicitly:

$$\frac{\partial F}{\partial y'} = \text{const}$$

- Case that F does not contain x explicitly:

$$F - y' \frac{\partial F}{\partial y'} = \text{const}$$

- Variable endpoint at $x = b$:

$$\left. \frac{\partial F}{\partial y'} \right|_{x=b} = 0$$

- Closure:

$$X, Y \in \mathcal{G} \;\Rightarrow\; X \bullet Y \in \mathcal{G}$$

- Identity element:

$$I \bullet X = X = X \bullet I$$

- Inverse:

$$X^{-1} \bullet X = I = X \bullet X^{-1}$$

- Abelian property:

$$Y \bullet X = X \bullet Y$$

- Coset relation:

$$X \sim Y \quad \text{if} \quad X^{-1}Y \in \mathcal{H}$$

- Conjugacy relation:

$$X \sim Y \quad \text{if} \quad \exists G_i \in \mathcal{G} \text{ s.t. } Y = G_i^{-1} X G_i$$

- Defining properties of representations:

$$\mathsf{D}(I) = \mathsf{I}_n\,, \qquad \mathsf{D}(X)\mathsf{D}(Y) = \mathsf{D}(XY)$$

- Equivalent representations:

$$\mathsf{E}(X) = \mathsf{S}^{-1}\mathsf{D}(X)\mathsf{S}$$

- Reducible representation:

$$\mathsf{D} = m_1\hat{\mathsf{D}}^{(1)} \oplus m_2\hat{\mathsf{D}}^{(2)} \oplus \cdots \oplus m_N\hat{\mathsf{D}}^{(N)}$$

- Characters of a representation:

$$\chi_{\mathsf{D}} = \{\operatorname{Tr}\mathsf{D}(X) \mid X \in \mathcal{G}\}$$

- Summation rule for irreps:

$$\sum_{i=1}^{N} n_i^2 = g$$

- Rotation of coordinate axes:

$$x_i' = L_{ij}x_j \quad \text{or} \quad \mathsf{x}' = \mathsf{L}\mathsf{x}$$
$$x_i = L_{ji}x_j' \quad \text{or} \quad \mathsf{x} = \mathsf{L}^{\mathrm{T}}\mathsf{x}'$$

- Orthogonality condition:

$$L_{ik}L_{jk} = \delta_{ij} \quad \text{or} \quad \mathsf{L}\mathsf{L}^{\mathrm{T}} = \mathsf{I}$$
$$L_{ki}L_{kj} = \delta_{ij} \quad \text{or} \quad \mathsf{L}^{\mathrm{T}}\mathsf{L} = \mathsf{I}$$

- Determinant condition for proper rotations:

$$\det \mathsf{L} = 1$$

- Transformation law for a general-order tensor:

$$T'_{ij\cdots k} = L_{ip}L_{jq}\cdots L_{kr}T_{pq\cdots r}$$
$$T_{ij\cdots k} = L_{pi}L_{qj}\cdots L_{rk}T'_{pq\cdots r}$$

- Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

- Levi–Civita symbol:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } i,j,k \text{ is an even perm of } 1,2,3 \\ -1 & \text{if } i,j,k \text{ is an odd perm of } 1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

- Determinant of a 3×3 matrix:

$$\det(\mathsf{A})\,\epsilon_{lmn} = A_{li}A_{mj}A_{nk}\epsilon_{ijk}$$