NATIONAL UNIVERSITY OF SINGAPORE

PC4130 Quantum Mechanics III

(Semester I: AY 2013-14)

Name of Examiner: Assoc. Prof. Gong Jiangbin

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name.**
2. This examination paper contains **four** questions and comprises **three** printed pages.
3. Answer ALL four questions.
4. Please start each question on a new page.
5. This is a CLOSED BOOK assessment.
6. One Help Sheet (A4 size, both sides) is allowed for this examination.

Question 1.

Briefly explain all the following items.

(a) WKB quantization rule

(b) Lippmann-Schwinger equation in scattering theory

(c) Heisenberg representation

(d) Einstein A and B coefficients

(e) Fine structure of hydrogen atom

Question 2.

An unperturbed quantum system is described by a simple Hamiltonian H^0, which has only three eigenstates. The first two are degenerate, with the common eigenvalue E_a^0, and the third state has a different eigenvalue E_b^0. Consider now the effects of a perturbation $\hat{V}$. In the presence of the perturbation, the full Hamiltonian in representation of H^0-eigenstates is given by the following matrix:

$$H = H^0 + \hat{V} = \begin{bmatrix} E_a^0 & 0 & 0 \\ 0 & E_a^0 & 0 \\ 0 & 0 & E_b^0 \end{bmatrix} + \begin{bmatrix} 0 & C & A \\ C & 0 & B \\ A & B & 0 \end{bmatrix}, \qquad (1)$$

where A, B, and C are small and real matrix elements arising from the perturbation.

(a) Using the non-degenerate perturbation theory, find the first-order and second-order corrections to the non-degenerate eigenvalue, as well as the first-order correction to the associated eigenfunction.

(b) Using the degenerate perturbation theory, find the correct zeroth-order eigenstates and the first-order corrections to the energy eigenvalue.

(c) Consider now a situation in which the perturbation $\hat{V}$ is suddenly switched on. If the system is initially prepared on one of the two degenerate states, find the time-dependence of the population on the nondegenerate state (you may directly use first-order time-dependent perturbation theory).

Question 3.

Using variational principle calculations, estimate the ground state energy of a particle (of mass m) moving in a one-dimensional potential $V(z)$ for $z \geq 0$, where $V(z) = +\infty$ for $z = 0$ and $V(z) = cz$ for $z > 0$, with $c > 0$. This kind of potential describes a ball bouncing on a hard floor in the presence of gravity. You may need the following integral:

$$\int_0^{+\infty} \exp(-ax^2)x^{2n}dx = \sqrt{\frac{\pi}{4a}}\left(\frac{1}{4a}\right)^n \frac{(2n)!}{n!}.$$

(a) The suggested trial wavefunction is $Az\exp(-bz^2)$. Justify this trial wavefunction.

(b) What is your estimate for a fixed b and what is your best prediction by optimizing b ?

Question 4.

Consider the adiabatic evolution of a quantum system with a time-dependent Hamiltonian $\hat{H}[\lambda(t)]$, where $\lambda(t)$ is the adiabatic parameter varying slowly with time t. The instantaneous eigenstates of $\hat{H}[\lambda(t)]$ are given by $|\psi_n[\lambda(t)]\rangle$, with nondegenerate eigenvalues $E_n[\lambda(t)]$.

(a) At time $t = 0$ the system is prepared on the eigenstate $|\psi_n[\lambda(0)]\rangle$ of $\hat{H}[\lambda(0)]$. Under the adiabatic approximation, write down the time-evolving wavefunction of the system with explicit expressions of a dynamical phase and a geometric phase.

(b) Based on the result in (a), construct the unitary transformation operator $\hat{U}(t, 0)$ (under the adiabatic approximation) that describes the mapping from an arbitrary initial state to a final state at time t.

(c) If the unitary operator $\hat{U}(t, 0)$ constructed in (b) is an *exact* propagator associated with a Hamiltonian $\hat{H}'(t)$, what is the expression of $\hat{H}'(t)$ in terms of $U(t, 0)$ and $\frac{\partial U(t,0)}{\partial t}$?

(d) Show that $\left(\hat{H}'(t) - \hat{H}[\lambda(t)]\right)$ is proportional to $\frac{d\lambda(t)}{dt}$, namely, the rate of change in $\lambda(t)$.

END OF PAPER (JG)