Question I (1)

$$\hat{H} = \begin{pmatrix} E_1 & \hbar\Omega_p \cos\omega_p t & 0 \\ \hbar\Omega_p^* \cos\omega_p t & E_2 & \hbar\Omega_s \cos\omega_s t \\ 0 & \hbar\Omega_s^* \cos\omega_s t & E_3 \end{pmatrix}$$

Using RWA, $\Omega\cos\omega t \approx \frac{1}{2}\Omega e^{i\omega t}$. the matrix simplifies to

$$\hat{H} = \begin{pmatrix} E_1 & \frac{1}{2}\hbar\Omega_p e^{i\omega_p t} & 0 \\ \frac{1}{2}\hbar\Omega_p^* e^{-i\omega_p t} & E_2 & \frac{1}{2}\hbar\Omega_s e^{i\omega_s t} \\ 0 & \frac{1}{2}\hbar\Omega_s^* e^{-i\omega_s t} & E_3 \end{pmatrix}$$

Question I (2)

Using the time dependent Schrödinger equation,

$$i\hbar\begin{pmatrix} \dot{C}_1 \\ \dot{C}_2 \\ \dot{C}_3 \end{pmatrix} = \begin{pmatrix} E_1 & \frac{1}{2}\hbar\Omega_p e^{i\omega_p t} & 0 \\ \frac{1}{2}\hbar\Omega_p^* e^{-i\omega_p t} & E_2 & \frac{1}{2}\hbar\Omega_s e^{i\omega_s t} \\ 0 & \frac{1}{2}\hbar\Omega_s^* e^{-i\omega_s t} & E_3 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

$$\tilde{C}_1 = C_1 e^{-i\omega_p t} \quad \Rightarrow \quad \dot{\tilde{C}}_1 = (\dot{C}_1 - i\omega_p C_1)e^{-i\omega_p t}$$
$$\tilde{C}_2 = C_2 \quad \Rightarrow \quad \dot{\tilde{C}}_2 = \dot{C}_2$$
$$\tilde{C}_3 = C_3 e^{i\omega_s t} \quad \Rightarrow \quad \dot{\tilde{C}}_3 = (\dot{C}_3 + i\omega_s C_3)e^{i\omega_s t}$$

$$i\hbar\dot{\tilde{C}}_1 = \left(E_1 C_1 + \frac{1}{2}\hbar\,\Omega_p e^{i\omega_p t} C_2 + \hbar\omega_p C_1\right)e^{-i\omega_p t}$$
$$i\hbar\dot{\tilde{C}}_2 = \frac{1}{2}C_1\hbar\Omega_p^* e^{-i\omega_p t} + C_2 E_2 + \frac{1}{2}\hbar\Omega_s e^{i\omega_s t}$$
$$i\hbar\dot{\tilde{C}}_3 = \left(\frac{1}{2}\hbar\Omega_s^* e^{-i\omega_s t} C_2 + C_3 E_3 - \hbar\omega_s C_3\right)e^{i\omega_s t}$$

$$\therefore\ i\hbar\begin{pmatrix} \dot{\tilde{C}}_1 \\ \dot{\tilde{C}}_2 \\ \dot{\tilde{C}}_3 \end{pmatrix} = \begin{pmatrix} E_1 + \hbar\omega_p & \frac{1}{2}\hbar\Omega_p & 0 \\ \frac{1}{2}\hbar\Omega_p^* & E_2 & \frac{1}{2}\hbar\Omega_s \\ 0 & \frac{1}{2}\hbar\Omega_s^* & E_3 - \hbar\omega_s \end{pmatrix}\begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \\ \tilde{C}_3 \end{pmatrix} \approx \underbrace{\begin{pmatrix} E_2 & \frac{1}{2}\hbar\Omega_p & 0 \\ \frac{1}{2}\hbar\Omega_p^* & E_2 & \frac{1}{2}\hbar\Omega_s \\ 0 & \frac{1}{2}\hbar\Omega_s^* & E_2 \end{pmatrix}}_{\text{our new static Hamiltonian}}\begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \\ \tilde{C}_3 \end{pmatrix}$$

Question II (1)

$$f(\theta,\phi) = -\frac{2m}{\hbar^2\kappa}\int_0^\infty r\sin\kappa r\, V(r)\, dr$$
$$= \frac{2mV_0}{\hbar^2\kappa}\int_0^a r\, sin\kappa r\, dr$$
$$= \frac{2mV_0}{\hbar^2\kappa}\left(\left[-\frac{r}{\kappa}\cos\kappa r\right]_0^a + \int_0^a \frac{\cos\kappa r}{\kappa}dr\right)$$
$$= \frac{2mV_0}{\hbar^2\kappa}\left(-\frac{a}{\kappa}\cos\kappa a + \left[\frac{\sin\kappa r}{\kappa^2}\right]_0^a\right)$$
$$= \frac{2mV_0}{\hbar^2\kappa^2}\left(\frac{1}{\kappa}\sin\kappa r - a\cos\kappa a\right)$$
$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 = \frac{4m^2V_0^2}{\hbar^4\kappa^4}\left(\frac{1}{\kappa}\sin\kappa r - a\cos\kappa a\right)^2$$

Question II (2)

The optical theorem is a general law of wave scattering theory, it relates the total cross-section to the imaginary part of the forward scattering amplitude. It has the formula

$$\sigma_{tot} = \frac{4\pi}{\kappa}Im[f(0)]$$

where $f(0)$ is the scattering amplitude at $\theta = 0$. It is derived using the conservation of probabilities in the derivations of the partial wave analysis. It is applicable for both elastic & inelastic scattering.

Question II (3)

$\sigma_{tot} = 0$, which cannot be true! The results don't make sense because the theorem is derived vigorously, but the Born approximation used here was only 1st order.

Question III (1)

The selection rules say $\Delta m = \pm 1,0$ and $\Delta l = \pm 1$. Since it iss in the ground state, we have $n = 1, l = 0, m = 0$. It can move to states like $|200\rangle, |210\rangle, |21-1\rangle, |211\rangle$.

$\therefore l = 1, m = -1,0,1.$

Question III (2)

$$E(t) = E_0\delta(t)\hat{z} \quad \Rightarrow \quad V = E_0 zq\delta(t)$$
$$P_{n\leftarrow 1} = \left|\langle\psi_n^0|\hat{U}_I(t)|\psi_1^0\rangle\right|^2$$
$$= \frac{1}{\hbar^2}\left|\int_0^t \langle\psi_n^0|E_{oz\delta(t)q}|\psi_1^0\rangle\, e^{\frac{i(E_n^0-E_1)t'}{\hbar}}\, dt'\right|^2$$
$$= \frac{|d_{n1}|^2}{\hbar^2}E_0^2\left|\int_0^t \delta(t)e^{\frac{i(E_n^0-E_1)t'}{\hbar}}\, dt'\right|^2$$
$$= \frac{|d_{n1}|^2}{\hbar^2}E_0^2$$

Question IV

Using the Gaussian wavefunction, $\left(\frac{2b}{\pi}\right)^{\frac{1}{4}} e^{-bx^2}$

$$\langle\hat{H}\rangle = \frac{\hbar^2 b}{2m} + \sqrt{\frac{2b}{\pi}}\int_{-\infty}^{\infty} cx^4 e^{-2bx^2}\, dx$$

$$\sqrt{\frac{2b}{\pi}}\int_{-\infty}^{\infty} cx^4 e^{-2bx^2}\, dx = 2\sqrt{\frac{2bc^2}{\pi}}\left[\sqrt{\frac{\pi}{8b}}\left(\frac{1}{8b}\right)^2 \frac{4!}{2!}\right] = \frac{3c}{16b^2}$$

$$\langle\hat{H}\rangle = \frac{\hbar^2 b}{2m} + \frac{3c}{16b^2}$$

$$\frac{d\langle\hat{H}\rangle}{db} = \frac{\hbar^2}{2m} - \frac{3c}{8b^3} = 0, \qquad \Rightarrow \quad b = \left(\frac{3mc}{4\hbar^2}\right)^{\frac{1}{3}}$$

∴ the expected ground state energy,

$$\langle\hat{H}\rangle = \frac{\hbar^2}{2m}\left(\frac{3mc}{4\hbar^2}\right)^{\frac{1}{3}} + \frac{3c}{16}\left(\frac{3mc}{4\hbar^2}\right)^{-\frac{2}{3}} = \frac{3}{4}\left(\frac{3\hbar^4 c}{4m^2}\right)^{\frac{1}{3}}$$

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