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Question 1 (a)

Using a Gaussian wavefunction $\psi(x) = \left(\frac{2b}{\pi}\right)^{\frac{1}{4}} e^{-bx^2}$,

$$\langle \hat{H} \rangle = \frac{\hbar^2 b}{2m} + \int_{-\infty}^{\infty} \sqrt{\frac{2b}{\pi}} e^{-2bx^2} \alpha |x| \, dx$$

$$= \frac{\hbar^2 b}{2m} + 2\int_{0}^{\infty} \sqrt{\frac{2b}{\pi}} \alpha x e^{-bx^2} \, dx$$

$$= \frac{\hbar^2 b}{2m} + 2\sqrt{\frac{2b}{\pi}} \left[\frac{e^{-2bx^2}}{-4b}\right]_0^{\infty}$$

$$= \frac{\hbar^2 b}{2m} + \alpha \sqrt{\frac{1}{2\pi b}}$$

$$\frac{d\langle \hat{H} \rangle}{db} = \frac{\hbar^2}{2m} + \frac{\alpha}{\sqrt{2\pi}} \left(\frac{1}{2}\right) b^{-\frac{3}{2}} = 0$$

$$b = \left(\frac{\alpha^2 m^2}{2\pi \hbar^4}\right)^{\frac{1}{3}}$$

∴ the ground state energy level,

$$\langle \hat{H} \rangle_{gs} = \frac{\hbar^2}{2m} \left(\frac{\alpha^2 m^2}{2\pi \hbar^4}\right)^{\frac{1}{3}} + \alpha \sqrt{\frac{1}{2\pi}} \left(\frac{\alpha^2 m^2}{2\pi \hbar^4}\right)^{-\frac{1}{6}} = \frac{3}{2} \left(\frac{\hbar^2 \alpha^2}{2\pi m}\right)^{\frac{1}{3}}$$

Question 1 (b)

Question 2 (a)

$$\begin{vmatrix} \cos\theta - E & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - E \end{vmatrix} = E^2 - \cos^2\theta - \sin^2\theta = 0, \qquad E = \pm 1$$

$E = 1,$

$$-\frac{\gamma B\hbar}{2}\begin{pmatrix} \cos\theta - 1 & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$y(\cos\theta + 1) + x\sin\theta\, e^{i\phi} = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta + 1 \\ \sin\theta\, e^{i\phi} \end{pmatrix} = \begin{pmatrix} 2\cos^2\frac{\theta}{2} - 1 + 1 \\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} e^{i\phi} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

$$\therefore \chi_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}, \qquad \chi_- = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\phi} \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

Question 2 (b)

$$\gamma_- = i \oint \left\langle \chi_-(\theta,\phi) \middle| \frac{\partial \chi_-(\theta,\phi)}{\partial\phi} \right\rangle d\phi$$

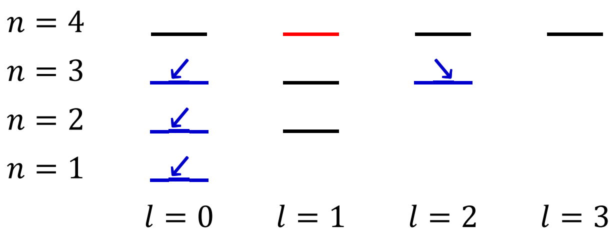
$$= \oint \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\phi} \\ -\cos\frac{\theta}{2} \end{pmatrix} \cdot \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\phi} \\ 0 \end{pmatrix} d\phi$$

$$= -\oint \sin^2\frac{\theta}{2}\, d\phi$$

$$= -2\pi \sin^2\frac{\theta}{2}$$

$$= \pi(\cos\theta - 1)$$

Question 3 (a)



∴ The possible final states,

$|322\rangle, |321\rangle, |320\rangle, |300\rangle, |200\rangle$ and $|100\rangle$

Question 3 (b)

Routes:

$|322\rangle \rightarrow |211\rangle \rightarrow |100\rangle$

$$|321\rangle \begin{matrix} \nearrow |211\rangle \searrow \\ \searrow |210\rangle \nearrow \end{matrix} |100\rangle$$

$$\begin{matrix} & \nearrow |211\rangle \searrow & \\ |320\rangle & \rightarrow |210\rangle \rightarrow & |100\rangle \\ & \searrow |21-1\rangle \nearrow & \end{matrix}$$

$$\begin{matrix} & \nearrow |211\rangle \searrow & \\ |300\rangle & \rightarrow |210\rangle \rightarrow & |100\rangle \\ & \searrow |21-1\rangle \nearrow & \end{matrix}$$

The state $|200\rangle$ will not reach the ground state.

Question 3 (c)

$$A = \frac{\omega_{nm}^3 |d_{nm}|^2}{3\pi\epsilon_0 \hbar c^3} \propto \omega_{nm}^3 |d_{nm}|^2 \propto \mu^3 \frac{1}{\mu_2} \propto \mu$$

The spontaneous rate of emission depends on μ, the reduced mass of hydrogen / deuterium. We find that $\mu_D > \mu_H$, since

$$\frac{m_e m_D}{m_e + m_D} \approx m_e\left(1 - \frac{m_e}{m_D}\right) > m_e\left(1 - \frac{m_e}{m_H}\right)$$

∴ Deuterium has a higher emission rate.

Question 4

a) Lippmann-Schwinger equation, $|\psi^{\pm}\rangle = G_0^{\pm}\hat{V}|\psi^{\pm}\rangle + |\psi_0\rangle$ is an equation postulated in the process of trying to seek time-independent eigenstates that match the boundary condition for a scattering problem, assuming structureless projectiles and targets (elastic collision). $G_0^{\pm}$ is the Green's function operator used to replace the singular operator $\frac{1}{E-\hat{H}_0}$, $G_0^{\pm} \approx \frac{1}{E-\hat{H}\pm i\epsilon}$. This equation proceeds to give us a coordinate representation of scattering solution.

b) The optical theorem, $\sigma_{tot} = \frac{4\pi}{k} Im[f(\theta = 0)]$, shows that the total cross section for a spherically symmetric scattering potential is exactly given by the imaginary part of the forward scattering amplitude multiplied by $\frac{4\pi}{k}$. This theorem can be extended to inelastic scattering cases as well, it is a very general law of wave scattering theory.

c) Interaction picture is an intermediate and very useful picture of quantum evolution. Given a Hamiltonian $\hat{H} = \hat{H}_0 + V(t)$ where $\hat{H}_0$ is time independent and $V(t)$ is a small time dependent perturbation, we factorize the exact unitary evolution (if it is known) into $U(t, t_0) = U_0(t, t_0)U_I(t, t_0)$, where U_0 is the evolution operator without perturbation and U_I is the additional correction due to the interaction. So in the Schrödinger picture kets evolve; in the Heisenberg picture the operators evolve; but in the interaction picture both kets and operators evolve. It can be used to derive the first order transition amplitudes between states.

d) The fine structure corrections are due to relativistic effects and spin-orbit coupling. The relativistic effects are taken into account by using the relativistic formula for kinetic energy, $T = \sqrt{p^2c^2 + m_e^2c^4} + m_ec^2$, which results in an extra $-\frac{p^4}{8m_e^3c^2}$ term in the Hamiltonian. The spin-orbit coupling came about due to the fact that a spinning electron has a magnetic dipole momentum along its spin angular momentum. This configuration results in an effective magnetic field along the direction of the orbital angular momentum. The Hamiltonian is then perturbed by $\hat{V}_{so} = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m_e^2c^2r^3}(\hat{S}\cdot\hat{L})$. This effect explains the splitting of the red H-alpha line. These two effects can be solved using time independent perturbation theory, and result in an energy change of $\Delta E_{fs}(nlj) = \frac{(E_{nlm}^0)^2}{2m_ec^2}\left(3 - \frac{4}{j+\frac{1}{2}}\right)$.

e) Fermi's Golden Rule, $W = \frac{dP_{tot}}{dt} = \frac{2\pi}{\hbar}|v_{nm}|^2\rho(E_m + \hbar\omega)$ is derived from 1st-order time dependent perturbation theory under the assumption that the transition matrix elements and the density of states are slowly varying functions. The rule shows that the rate of transition to a continuum of final states is proportional to the square of the transition matrix element $|v_{nm}|^2$ between the initial and final states. It is also proportional to the density of states $\rho(E_m + \hbar\omega)$, evaluated at an energy that differs from the initial state energy by $\hbar\omega$, where ω is the driving frequency of a periodic perturbation.

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