Question 1 (i)

It is an ideal gas, it has N identical monatomic particles, and they are indistinguishable.

$$z = V\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}}$$

$$Z_N = \frac{z^N}{N!} = \frac{V^N}{N!}\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3N}{2}}$$

$$S = k\left(\ln Z_N - \beta\frac{\partial}{\partial\beta}\ln Z_N\right) = Nk\left[\ln\frac{V}{N} + \frac{3}{2}\ln\left(\frac{2\pi m}{h^2\beta}\right) + \frac{5}{2}\right] \quad \text{[shown]}$$

Hint: expand and use Sterling's Formula.

Question 1(ii)

Take the formula of entropy S, since the gases have constant density, we see that the only difference in S for two different gases is the term $\frac{3}{2}\ln\left(\frac{2\pi m}{h^2\beta}\right)$, where there is a difference in m. So if the gases are different there is net change in entropy, but if the gases are different, there isn't. This is the entropy of mixing, and also part of the Gibbs Paradox.

Question 2 (i)

Classical ideal gas, so

$$Z_N = \frac{1}{N!}\frac{1}{h^{3N}}\int e^{-\frac{\beta p^2}{2m}}\, d^{3N}p\, d^{3N}r = \frac{V^N}{N!}\frac{1}{h^{3N}}\left[\int_{-\infty}^{\infty} e^{-\frac{\beta p^2}{2m}}\, dp\right]^{3N} = \frac{V^N}{N!}\left(\frac{1}{\lambda^3}\right)^N \quad \text{[shown]}$$

$$P = \frac{1}{\beta}\left(\frac{\partial}{\partial V}\ln Z_N\right) = \frac{1}{\beta}\frac{N}{V} = \frac{NkT}{V}$$

Hint: use the integral formula given.

Question 2 (ii)

$$\mathcal{Z} = \sum_s e^{-\frac{\beta p^2}{2m}+\beta\mu N} = \sum^N Z_N\zeta^N = \sum^N \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N \zeta^N = e^{\frac{\partial V}{\lambda^3}}$$

$$P = \frac{1}{\beta}\frac{\partial}{\partial V}(\ln Z_N)_{\beta,\beta\mu} = \frac{1}{\beta}\frac{\partial}{\partial V}\left(\frac{\partial V}{\lambda^3}\right) = \frac{NkT}{V}$$

Hint: Use the Maclaurin expansion of the exponent e, and the fact that $\frac{N}{V} = \frac{\partial}{\lambda^3}$.

Question 2 (iii)

For Z_N,

$$U = -\frac{\partial}{\partial\beta}(\ln Z_N) = -\frac{\partial}{\partial\beta}\left(-\ln\beta^{\frac{3N}{2}}\right) = \frac{3}{2}NkT$$

For $\mathcal{Z}_N$,

$$U = -\frac{\partial}{\partial\beta}(\ln \mathcal{Z}_N) = -\frac{\partial}{\partial\beta}\left[{}_3V\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}}\right] = \frac{3}{2}NkT$$

Question 3 (i)

N localized distinguishable dipoles, so

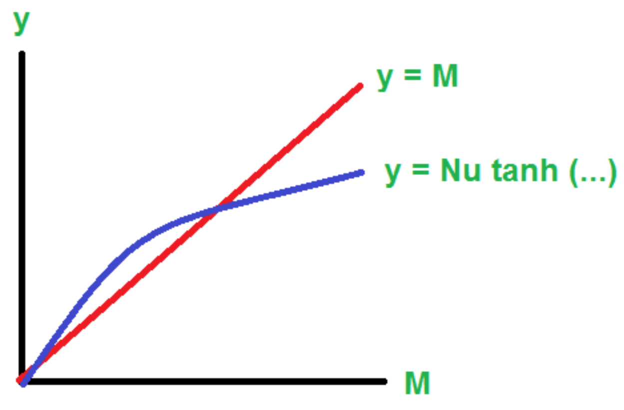
$$Z_N = \left[\sum_r e^{\beta E_r}\right]^N = \left(e^{\beta\mu H} + e^{-\beta\mu H}\right)^N = 2^N\cosh^N\beta\mu H$$

$$M = -\frac{U}{H} = \frac{1}{H}\frac{\partial}{\partial\beta}(\ln Z_N) = \frac{2N}{H}\frac{\partial}{\partial\beta}[\ln(\cosh\beta\mu H)] = N\mu\tanh\beta\mu H \quad \text{[shown]}$$

Question 3 (ii)

$$H = H_a + \lambda M$$
$$M = N\mu\tanh\beta\mu(H_a + \lambda M)$$

Graphically, it looks like this:



Question 3 (iii)

When $H_a = 0$, $M = N\mu\tanh\beta\mu\lambda M$. We have 2 solutions, either $M = 0$ or $M \neq 0$.

If $\frac{d}{dM}(N\mu\tanh\beta\mu\lambda M) > 1$ near $M = 0$ (near the critical temperature), then

$$N\mu\tanh\beta\mu\lambda M \approx N\mu(\beta\mu\lambda M)$$
$$\frac{d}{dM}(N\mu\tanh\beta\mu\lambda M) \approx N\mu^2\beta\lambda$$

$$N\mu^2\beta\lambda > 1$$
$$\frac{N\mu^2\lambda}{kT_c} > 1$$
$$T_c = \frac{N\mu^2\lambda}{k} \quad \text{[shown]}$$

Hint: $\tanh x \approx x$ for very small x.

Question 3 (iv)

$$M = N\mu\left[\beta\mu\lambda M - \frac{(\beta\mu\lambda M)^3}{3}\right]$$
$$1 = \frac{N\mu^2\lambda}{kT} - \frac{N\mu^4\lambda^3}{3k^3T^3}M^2$$

Substituting from 3(iii), we get

$$\left(\frac{M}{N\mu}\right)^2 = \frac{3T^3}{T_c^3}\left(\frac{T_c}{T} - 1\right)$$
$$\frac{M}{N\mu} = \sqrt{3}\left(\frac{T}{T_c}\right)^{\frac{3}{2}}\left(\frac{T_c - T}{T}\right)^{\frac{1}{2}}$$

Hint: in the first step, let $\tanh x \approx x - \frac{x^3}{3}$.

Question 3 (v)

$$M \approx N\mu\left[\beta\mu(H + \lambda M) - \frac{(\beta\mu)^3(H+\lambda M)^3}{3}\right] \approx \frac{N\mu^2}{kT}(H + \lambda M) = \frac{N\mu^2}{k}\frac{1}{T - T_c}H$$
$$\chi = \frac{\partial M}{\partial H} = \frac{N\mu^2}{k}\frac{1}{T - T_c}$$

In accordance to Curie-Weiss Law.

Question 4 (i)

$$\lambda^3 n = g_{\frac{3}{2}}(\zeta)$$
$$\frac{\lambda^3 P}{kT} = g_{\frac{5}{2}}(\zeta)$$

For condensation to occur, $n\lambda^3 \geq g_{\frac{3}{2}}(1)$. At the condensation point,

$$n\lambda^3 = n\left(\frac{h^2}{2\pi mkT}\right)^{\frac{3}{2}} = g_{\frac{3}{2}}(1)$$

And this gives a boundary line between the gas and the condensed phase.

At fixed n,

$$n^{\frac{2}{3}}\left(\frac{h^2}{2\pi mkT_c}\right) = g_{\frac{3}{2}}(1)^{\frac{2}{3}}$$

$$T_c = \frac{2\pi\hbar^2}{mk}\left[\frac{n}{g_{\frac{3}{2}}(1)}\right]^{\frac{2}{3}}$$

Similarly at fixed T,

$$n_c = \left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}} g_{\frac{3}{2}}(1) \quad \text{[shown]}$$

Hint: condensation occurs when $n > n_c$ and $T < T_c$.

Question 4 (ii)

The condensate consists of n_0 particles in the zero-momentum state below T_c. So

$$N = n_0 + \frac{V}{\lambda^3} g_{\frac{3}{2}}(1)$$

$$\frac{n_0}{N} = 1 - \frac{g_{\frac{3}{2}}(1)}{n\lambda^3} = 1 - \left(\frac{T}{T_c}\right)^{\frac{3}{2}}$$

Question 4 (iii)

$\frac{P}{kT} = \frac{g_{\frac{5}{2}}(1)}{\lambda^3}$ in condensation phase. So

$$P = kT g_{\frac{5}{2}}(1)\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}$$

$$\frac{dP}{dT} = \frac{5}{2}k\frac{g_{\frac{5}{2}}(1)}{\lambda^3} \quad \text{[shown]}$$

In Clausius-Clapeyron equation,

$$\frac{dP}{dT} = \frac{\Delta s}{\Delta v} = \frac{L}{T\Delta v}$$

$$\Delta v = \frac{1}{n_c} = \frac{\lambda^3}{g_{\frac{3}{2}}(1)}$$

Rewriting the equation,

$$\frac{dP}{dT} = \frac{5}{2} k \frac{g_{\frac{5}{2}}(1)}{g_{\frac{3}{2}}(1)} \frac{1}{\Delta v}$$

Change of specific entropy,

$$\Delta s = \frac{L}{T} = \frac{5}{2} k \frac{g_{\frac{5}{2}}(1)}{g_{\frac{3}{2}}(1)}$$

$$\therefore L = \frac{5}{2} \frac{g_{\frac{5}{2}}(1)}{g_{\frac{3}{2}}(1)} kT$$

Solutions provided by:
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