National University of Singapore

PC4241 Statistical Mechanics

(Semester I: AY2008-09, 29 November)

Time Allowed: Two Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR** questions and comprises **FOUR** printed pages.
2. All questions carry equal marks.
3. Answer any **THREE** questions.
4. This is a CLOSED BOOK examination.
5. One A4 double-sided sheet of formulae and equations is allowed.

1. (i) The single molecules partition function z for an ideal gas consisting of N identical monatomic molecules, each of mass m, enclosed in a compartment of volume V is given by

$$z = V\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}}.$$

Assume the molecules are indistinguishable. Show that the entropy S of the gas is given by

$$S = Nk\left[\ln\frac{V}{N} + \frac{3}{2}\ln\left(\frac{2\pi m}{h^2\beta}\right) + \frac{5}{2}\right].$$

(ii) Consider two gases of the same temperature and same number density originally in adjoining compartments separated by a partition. The partition is then withdrawn to allow the gases to mix. Determine the entropy of mixing for the case of different gases and for the case of same gases. Comment on the results.

2. (i) Show that the partition function Z_N for a classical ideal gas is given by

$$Z_N = \frac{V^N}{N!}\left(\frac{1}{\lambda^3}\right)^N,$$

where $\frac{1}{\lambda} = \left(\frac{2\pi mkT}{h^2}\right)^{\frac{1}{2}}$.

Hence, obtain the pressure of the gas as a function of N, V and T.

(ii) Show that the grand partition function $\mathcal{Z}_N$ for the gas is given by

$$\mathcal{Z}_N = e^{zV/\lambda^3},$$

where z is the fugacity.

Hence, obtain the pressure of the gas as a function of N, V and T.

(iii) Obtain the energy of the gas as a function of N and T from both Z_N and $\mathcal{Z}_N$.

$$\left[\int_{-\infty}^{\infty} dx e^{-x^2} = \pi^{\frac{1}{2}}\right]$$

3. Consider a solid containing N dipoles localized at lattice sites. Each dipole, with a magnetic moment μ, in a magnetic field H can exist in only two states with energy $\pm\mu H$.

(i) Show that the partition function Z_N and the total magnetic moment M of the solid are given respectively by

$$Z_N = [2\cosh\beta\mu H]^N,$$

$$M = N\mu\tanh\beta\mu H.$$

(ii) Assume that H consists of an external applied field H_a, plus a local field $H_l = \lambda M$. Write the self-consistence condition that determines M.

(iii) Show that there is spontaneous magnetization for $H_a = 0$ below a critical temperature T_c given by

$$T_c = \frac{N\mu^2\lambda}{k}.$$

(iv) For $H_a = 0$, show that the magnetization just below T_c behaves like

$$\frac{M}{N\mu} \approx \sqrt{3}\left(\frac{T}{T_c}\right)^{\frac{3}{2}}\left(\frac{T_c - T}{T}\right)^{\frac{1}{2}}$$

(v) Show that the magnetic susceptibility $\chi = \frac{\partial M}{\partial H}$ diverges as $T \to T_c$ from above, in accordance with the Curie-Weiss law

$$\chi = \frac{N\mu^2}{k(T - T_c)}.$$

$$\left[\tanh x \approx x - \frac{x^3}{3} \quad \text{for } x \ll 1\right]$$

4. The parametric equations of state for a gas of bosons can be written as

$$\lambda^3 n = g_{3/2}(z),$$

$$\frac{\lambda^3 P}{kT} = g_{5/2}(z),$$

where λ is the thermal wavelength, z the fugacity and $g_{3/2}(z)$ and $g_{5/2}(z)$ are the Bose functions.

(i) Show that , below a certain temperature T_c at fixed n and above a certain number density n_c at fixed T, a condensation occurs. Show that T_c and n_c are given respectively by

$$T_c = \frac{2\pi\hbar^2}{mk}\left[\frac{n}{g_{3/2}(1)}\right]^{\frac{2}{3}},$$

$$n_c = \left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}} g_{3/2}(1).$$

(ii) Show how the number of bosons in the ground state varies as a function of temperature below T_c.

(iii) In the condensation, show that

$$\frac{dP}{dT} = \frac{5}{2}k\frac{g_{5/2}(1)}{\lambda^3}.$$

Hence, show that the condensation satisfies the Clausius-Clapeyron equation. Determine the change of specific entropy and the latent heat of transition.

$$\left[g_k = \sum_{l=1}^{\infty}\frac{z^l}{l^k}, \qquad z\frac{d}{dz}g_k(z) = g_{k-1}(z)\right]$$

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