Question 1 (i)

So we have 2D, N atoms, and 2N independent harmonic oscillators.

$$Z_1 = \sum^{n} e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)} = e^{-\frac{\beta\hbar\omega}{2}} \sum^{n} e^{-\beta\hbar\omega n} = \frac{1}{2}\operatorname{csch}\frac{\beta\hbar\omega}{2}$$

$$Z_{2N} = (Z_1)^{2N} = \left(e^{\frac{\beta\hbar\omega}{2}} - e^{-\frac{\beta\hbar\omega}{2}}\right)^{-2N}$$

$$E = -\frac{\partial}{\partial\beta}\ln Z = 2N\frac{\partial}{\partial\beta}\left[\ln\left(2\sinh\frac{\beta\hbar\omega}{2}\right)\right] = N\hbar\omega\coth\frac{\beta\hbar\omega}{2}$$

Hint: be familiar with the conversions between hyperbolic and exponential functions.

Question 1 (ii)

$$C = \frac{\partial E}{\partial T} = -N\hbar\omega\operatorname{csch}^2\left(\frac{\beta\hbar\omega}{2}\right)\left(-\frac{\hbar\omega}{2kT^2}\right) = \frac{N\hbar^2\omega^2}{2kT^2}\operatorname{csch}^2\frac{\beta\hbar\omega}{2}$$

At high temperature,

$$C \approx \frac{N\hbar^2\omega^2}{2kT^2}\frac{4}{\beta^2\hbar^2\omega^2} = 2Nk$$

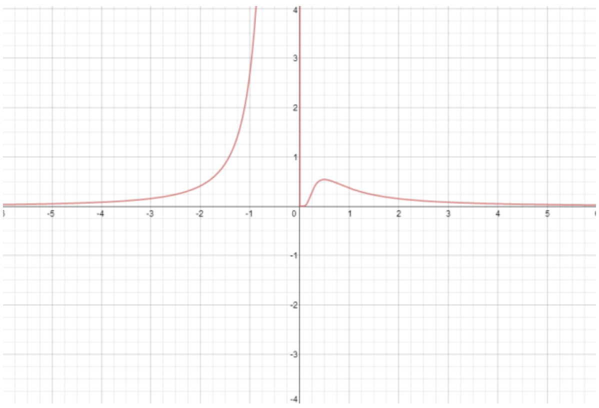
C is constant in accordance to Curie's Law.

At low temperature,

$$C = \frac{N\hbar^2\omega^2}{2kT^2}\frac{1}{\left(e^{\frac{\beta\hbar\omega}{2}} - e^{-\frac{\beta\hbar\omega}{2}}\right)^2} \approx \frac{N\hbar^2\omega^2}{2kT^2}e^{-\beta\hbar\omega} \approx 0$$

Hint 1: Curie's Law at high temperature includes the following 3 things: $M \propto \frac{1}{T}$, $\chi \propto \frac{1}{T}$ and $C \propto k$.

Hint 2: $e^{-x} \approx 0$ when x is big, and $e^{x} \approx 1 + x$ when x is close to zero. The graph of $y = \frac{e^{-\frac{1}{x}}}{x^2}$ looks something like the following, and it gets to zero as x tends to zero:



Question 1 (iii)

$$S = k\ln Z - \frac{U}{T} = 2Nk\ln\left(2\sinh\frac{\beta\hbar\omega}{2}\right) - \frac{N\hbar\omega}{T}\coth\frac{\beta\hbar\omega}{2}$$

For low temperature,

$$S \approx 2Nk\ln e^{\frac{\beta\hbar\omega}{2}} - \frac{N\hbar\omega}{T} = \frac{2Nk\hbar\omega}{2kT} - \frac{N\hbar\omega}{T} = 0$$

In accordance to the 3rd Law of Thermodynamics.

At high temperature,

$$S \approx 2Nk\ln\beta\hbar\omega - \frac{N\hbar\omega}{T}\frac{2}{\beta\hbar\omega} = 2Nk\left[\ln\left(\frac{\hbar\omega}{kT}\right) - 1\right]$$

Question 2 (i)

For $T \to 0$, $n_k \to \theta(\mu - \epsilon_k)$, where

$$\theta(\mu - \epsilon_k) = \begin{cases} 0, & \mu < \epsilon_k \\ 1, & \mu > \epsilon_k \end{cases}$$

$$N = \sum_{\epsilon<\epsilon_k} 1 = \frac{2A}{(2\pi)^2}\pi k_F^2 = \frac{A}{2\pi}k_F^2$$

$$k_F = \sqrt{\frac{2\pi N}{A}}$$

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(2\pi\frac{N}{A}\right) = \frac{\pi\hbar^2}{m}\frac{N}{A}$$

$$U = \sum_{k<k_F}\epsilon_k = \frac{2A}{(2\pi)^2}\int_0^{k_F}\frac{\hbar^2k^2}{2m}2\pi k\,dk = \frac{A\hbar^2k_F^4}{8m\pi} = \frac{1}{2}N\epsilon_F \quad \text{[shown]}$$

Hint: Try thinking of $\theta(\mu - \epsilon_k)$ as a 2D Dirac delta function, a circle on a 2D plane with the origin as its centre. Another thing here is, k_F is the wave factor, not to be confused with the Boltzmann constant!

Question 2 (ii)

$$N = \sum_r n_r = \frac{2A}{(2\pi)^2}\int\frac{2\pi k\,dk}{e^{\beta(\epsilon-\mu)}+1} = \frac{mA}{\pi\hbar^2}\int\frac{mA}{\pi\hbar^2}\frac{d\epsilon}{e^{\beta(\epsilon-\mu)}+1} = \int n(\epsilon)f(\epsilon)\,d\epsilon$$

So we can see that

$$f(\epsilon) = \frac{Am}{\pi\hbar^2} = G'(\epsilon), \qquad G(\epsilon) = \frac{Am}{\pi\hbar^2}\epsilon, \qquad G''(\epsilon) = 0$$

$$\therefore N = \frac{Am}{\pi\hbar^2}\mu$$

At $T = 0$,

$$N = \frac{mA}{\pi\hbar^2}\epsilon_F = \frac{mA}{\pi\hbar^2}kT_F$$

$\therefore \mu = kT_F = \epsilon_F$, it is independent of temperature.

Question 2 (iii)

$$U = \int n(\epsilon)f(\epsilon)\epsilon\, d\epsilon = \int n(\epsilon)G'(\epsilon)\, d\epsilon$$

$$G'(\epsilon) = \frac{Am}{\pi\hbar^2}\epsilon, \qquad G(\epsilon) = \frac{Am}{2\pi\hbar^2}\epsilon^2, \qquad G''(\epsilon) = \frac{Am}{\pi\hbar^2}$$

$$U = \frac{Am\mu^2}{2\pi\hbar^2} + \frac{\pi^2}{6}(kT)^2\frac{Am}{\pi\hbar^2} = \frac{1}{2}N\mu + \frac{N\pi^2}{6}\frac{(kT)^2}{\mu} = \frac{1}{2}NkT_F\left[1 + \frac{\pi^2}{3}\left(\frac{T}{T_F}\right)^2\right]$$

$$C_v = \frac{\partial U}{\partial T} = \frac{1}{2}NkT_F\left(\frac{2\pi^2 T}{3T_F^2}\right) = \frac{Nk\pi^2}{3}\frac{T}{T_F}$$

Question 3 (i)

$$n\lambda^3 = e^{\mu\beta} \mp \frac{e^{2\mu\beta}}{2^{\frac{3}{2}}} + \cdots$$

1st order approximation, $n\lambda^3 = e^{\mu\beta}$.

2nd order approximation, $n\lambda^3 = e^{\mu\beta} \mp \frac{(n\lambda^3)^2}{2^{\frac{3}{2}}}$

$$\Rightarrow e^{\mu\beta} = n\lambda^3\left(1 \pm \frac{n\lambda^3}{2^{\frac{3}{2}}}\right)$$

$$\mu = kT\left[\ln n\lambda^3 + \ln\left(1 \pm \frac{n\lambda^3}{2^{\frac{3}{2}}}\right)\right]$$

For high temperature, λ^3 is small, so $\ln\left(1 \pm \frac{n\lambda^3}{2^{\frac{3}{2}}}\right) \approx \pm\frac{n\lambda^3}{2^{\frac{3}{2}}}$, and

$$\therefore \mu = kT\left[\ln n\lambda^3 \pm \frac{n\lambda^3}{2^{\frac{3}{2}}} + \cdots\right] \quad \text{[shown]}$$

Question 3 (ii)

$$F = -kT \ln Z_N = -kT \ln \left[\frac{1}{N!} \left(\frac{V}{\lambda^3} \right)^N \right]$$

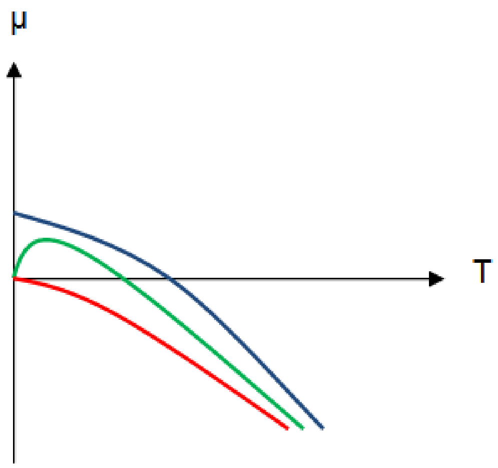
$$PV = \frac{V}{\beta} \frac{\partial}{\partial V} (\ln Z_N) = kTV \frac{\partial}{\partial V} \left\{ \ln \left[\frac{1}{N!} \left(\frac{V}{\lambda^3} \right)^N \right] \right\} = NkT$$

$$G = N\mu = F + PV = kT \left[\ln N! - N \ln \left(\frac{V}{\lambda^3} \right) + N \right] \approx NkT \left[\ln N - \ln \left(\frac{V}{\lambda^3} \right) \right]$$

$$\mu = kT \ln \left(\frac{N\lambda^3}{V} \right) = kT \ln(n\lambda^3) \quad \text{[shown]}$$

Hint: Use Stirling's formula for the simplification of G.

Question 3 (iii)



The fermions follow the blue line, the bosons the red line, while classical particles follow the green line. The intersection with the green line and the T axis is T_c. The quantum correction is the second term of the equation for μ. The results show that the chemical potential curve for fermions is higher than classical particles, and it starts from a non-zero value of $\mu = \epsilon_F$. The boson curve is lower than classical and is always negative.

Question 4 (i)

For the bose gas,

$$\frac{P\lambda^3}{kT} = g_{\frac{5}{2}}(\zeta), \qquad \frac{N\lambda^3}{V} = g_{\frac{3}{2}}(\zeta)$$

At $T > T_c$,

$$U = \frac{3}{2} PV = \frac{V}{\lambda^3} kT g_{\frac{5}{2}}(\zeta) = \frac{3}{2} \frac{kT}{\lambda^3} V g_{\frac{5}{2}}(\zeta)$$

$$\Omega_G = -PV = -\frac{kT}{\lambda^3} V g_{\frac{5}{2}}(\zeta)$$
$$G = N\mu = NkT \ln \zeta$$
$$F = \Omega_G + G = -\frac{kT}{\lambda^3} V g_{\frac{5}{2}}(\zeta) + NkT \ln \zeta$$
$$S = \frac{U}{T} - \frac{F}{T} = \frac{3}{2}\frac{k}{\lambda^3} V g_{\frac{5}{2}}(\zeta) + \frac{k}{\lambda^3} V g_{\frac{5}{2}}(\zeta) - Nk \ln \zeta = \frac{5}{2}\frac{kV}{\lambda^3} g_{\frac{5}{2}}(\zeta) - Nk \ln \zeta$$
$$C_v = \left(\frac{\partial U}{\partial T}\right)_{\beta\mu} = \frac{3}{2} kV \frac{\partial}{\partial T}\left[\frac{T}{\lambda^3} g_{\frac{5}{2}}(\zeta)\right] = \frac{5}{2}\frac{3}{2}\frac{kV}{\lambda^3} g_{\frac{5}{2}}(\zeta) + \frac{3}{2}\frac{kTV}{\lambda^3}\frac{\partial \zeta}{\partial T}\frac{\partial}{\partial \zeta} g_{\frac{5}{2}}(\zeta)$$
$$= \frac{15}{4}\frac{V}{\lambda^3} g_{\frac{5}{2}}(\zeta) - \frac{9}{4} Nk \frac{g_{\frac{3}{2}}(\zeta)}{g_{\frac{1}{2}}(\zeta)} \quad \text{[shown]}$$

Hint: to find the expression of C_v, use the relation $\frac{\partial \zeta}{\partial T} = -\frac{3N\lambda^3}{2TV}\frac{\zeta}{g_{\frac{1}{2}}(\zeta)}$, which can be obtained by partial differentiating both sides of the equation $\frac{N\lambda^3}{V} = g_{\frac{3}{2}}(\zeta)$ with respect to T.

Question 4 (ii)

For $T < T_c$,

$$\zeta = 1, \qquad g_{\frac{5}{2}}(1) = 1.342, \qquad g_{\frac{3}{2}}(1) = 2.612, \qquad g_{\frac{1}{2}}(1) = \infty$$

$$U = \frac{3}{2}\frac{kTV}{\lambda^3}(1.342),$$
$$\Omega_G = -\frac{1.342kTV}{\lambda^3}$$
$$G = 0,$$
$$F = \frac{1.342kTV}{\lambda^3} = -\Omega_G$$
$$S = \frac{5}{2}\frac{kV}{\lambda^3}(1.342)$$
$$C_v = \frac{15}{4}\frac{Vk}{\lambda^3}(1.342)$$

Question 4 (iii)

Since $T > T_c$,

$$C_v = \frac{15}{4}\frac{V}{\lambda^3} g_{\frac{5}{2}}(\zeta) - \frac{9}{4} Nk \frac{g_{\frac{3}{2}}(\zeta)}{g_{\frac{1}{2}}(\zeta)} = \frac{15}{4} Nk \frac{g_{\frac{5}{2}}(\zeta)}{g_{\frac{3}{2}}(\zeta)} - \frac{9}{4} Nk \frac{g_{\frac{3}{2}}(\zeta)}{g_{\frac{1}{2}}(\zeta)}$$

At $T \gg T_c$,

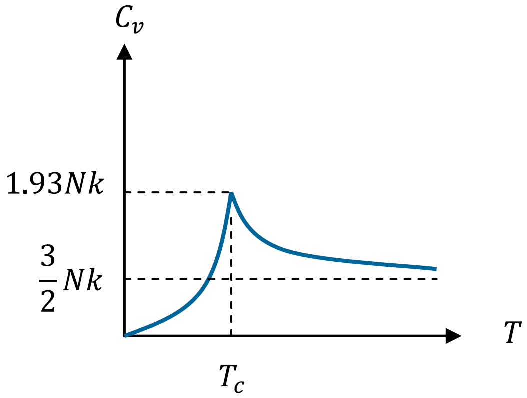
$$e^{\frac{\mu}{kT}} \to \frac{\mu}{kT}, \qquad g_k(\zeta) \approx \zeta \approx n\lambda^3$$

$$\frac{C_v}{Nk} \approx \frac{15}{4} - \frac{9}{4} = \frac{3}{2}$$

At $T = T_c, \zeta = 1$,

$$\frac{C_v}{Nk} = \frac{15}{4}\frac{1.342}{2.612} \approx 1.93$$

At $T = 0$, $C_v = 0$. So our graph look like this:



Hint: Just in case you are confused, here's a table to summarize what equations are valid at what temperatures. In simple words, the two equations are valid above T_c, $\zeta = 1$ at $T = T_c$, and the equation involving n is not valid below T_c!

Temperature condition	The two equations should be
$T > T_c$	$\frac{P\lambda^3}{kT} = g_{\frac{5}{2}}(\zeta)$, $\frac{N\lambda^3}{V} = g_{\frac{3}{2}}(\zeta)$
$T = T_c$	$\frac{P\lambda^3}{kT} = g_{\frac{5}{2}}(1)$, $\frac{N\lambda^3}{V} = g_{\frac{3}{2}}(1)$
$T < T_c$	$\frac{P\lambda^3}{kT} = g_{\frac{5}{2}}(1)$, $\frac{N\lambda^3}{V} > g_{\frac{3}{2}}(1)$

Solutions provided by:
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