National University of Singapore

PC4241 Statistical Mechanics

(Semester I: AY2010-11, 23 November)

Time Allowed: Two Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR** questions and comprises **FOUR** printed pages.
2. All questions carry equal marks.
3. Answer any **THREE** questions.
4. This is a CLOSED BOOK examination.
5. One A4 double-sided sheet of formulae and equations is allowed.

1. Graphene, a one-atom-thick planar sheet of carbon atoms, can be considered as a system of N atoms in a two-dimensional lattice, which is equivalent to $2N$ independent harmonic oscillations each with vibrational energy.

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad , \quad n = 0, 1, \ldots,$$

where ω is the vibrational angular frequency.

(i) Calculate the partition function and the energy of the system.

(ii) Calculate the heat capacity of the system and its high- and low-temperature limits. Comment on the results.

(iii) Calculate the entropy of the system and its high- and low-temperature limits. Comment on the result of the low-temperature limit.

2. It is experimentally possible to confine electrons to a two-dimensional plane. Consider a system of N free electrons, each of mass m, confined to a square of area A.

(i) Show that at temperature $T = 0K$, the Fermi energy ε_F, Fermi wavevector k_F and internal energy U of the system are given respectively by

$$\varepsilon_F = \frac{\pi\hbar^2}{m}\frac{N}{A} \quad , \quad k_F = \left(2\pi\frac{N}{A}\right)^{\frac{1}{2}} \quad , \quad U = \frac{1}{2}N\varepsilon_F \quad .$$

(ii) Obtain the density of state $f(\varepsilon)$ from (i) and find the expression for N using the Sommerfield expansion,

$$\int_0^\infty n(\varepsilon)G'(\varepsilon)d\varepsilon = G(\mu) + \frac{\pi^2}{6}(kT)^2 G''(\mu) \quad .$$

Hence, show that at low temperatures, the chemical potential μ, is independent of temperature.

(iii) Calculate the internal energy U and heat capacity C_V of the system at low temperatures as functions of N, T and ε_F.

3. (i) The fugacity z for quantum gases can be obtained by solving the equation

$$n\lambda^3 = z \mp \frac{z^2}{2^{\frac{3}{2}}} + \frac{z^3}{3^{\frac{3}{2}}} \mp \ldots,$$

where the upper sign refers to fermions and the lower sign to bosons, n is the density and λ the thermal wavelength. At high temperatures, show that the chemical potential μ is given by

$$\mu = kT\left[\ln\left(n\lambda^3\right) \pm \frac{n\lambda^3}{2^{\frac{3}{2}}} - \ldots\right] .$$

(ii) The partition function Z_N for a classical gas is given by

$$Z_N = \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N .$$

Use Z_N to obtain the Helmholtz free energy and the Gibbs free energy. Hence show that

$$\mu = kT\ln\left(n\lambda^3\right) .$$

(iii) Sketch the graphs of chemical potential vs temperature for the quantum and classical gases. Comment on your results.

4. The parametric equations of state for a Bose gas are given by

$$\frac{\lambda^3 P}{kT} = g_{\frac{5}{2}}(z) \ , \ \frac{\lambda^3 N}{V} = g_{\frac{3}{2}}(z)$$

where λ is the thermal wavelength, z the fugacity and the Bose function $g_k = \sum_{l=1}^{\infty} \frac{z^l}{l^k}$.

(i) Show that at temperature T higher than the Bose-Einstein condensation temperature T_c, the internal energy U, grand potential Ω_G, Gibbs free energy G, Helmholtz free energy F, entropy S and heat capacity C_V are given respectively by

$$U = \frac{3}{2}\frac{kT}{\lambda^3} V g_{\frac{5}{2}}(z) \quad ,$$

$$\Omega_G = -\frac{kT}{\lambda^3} V g_{\frac{5}{2}}(z) \quad ,$$

$$G = NkT \ln z \quad ,$$

$$F = -\frac{kT}{\lambda^3} V g_{\frac{5}{2}}(z) + NkT \ln z \quad ,$$

$$S = \frac{5}{2}\frac{k}{\lambda^3} V g_{\frac{5}{2}}(z) - Nk \ln z \quad ,$$

$$C_V = \frac{15}{4} Vk \frac{g_{\frac{5}{2}}(z)}{\lambda^3} - \frac{9}{4} Nk \frac{g_{\frac{3}{2}}(z)}{g_{\frac{1}{2}}(z)} \quad .$$

(ii) For temperature $T < T_c$ obtain the corresponding expressions for the thermodynamic functions in (i).

(iii) Calculate C_V / Nk for $T >> T_c$ and $T = T_c$ and find the temperature behavior of C_V near absolute zero. Sketch the graph of C_V vs T.

$$\left[g_{\frac{5}{2}}(1) = 1.342 \quad , \quad g_{\frac{3}{2}}(1) = 2.612 \right]$$

Ng S C

---End of Paper---